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Critical Points Inside the Gaps of Ground State Laminations for Some Models in Statistical Mechanics

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Abstract We consider models of interacting particles situated in the points of a discrete set Λ. The state of each particle is determined by a real variable. The particles are interacting with each other and we are interested in ground states and other critical points of the energy (metastable states).

Under the assumption that the set Λ and the interaction are symmetric under the action of a group G—which satisfies some mild assumptions—, that the interaction is ferromagnetic, as well as periodic under addition of integers, and that it decays with the distance fast enough, it was shown in a previous paper that there are many ground states that satisfy an order property called self-conforming or Birkhoff. Under some slightly stronger assumptions all ground states satisfy this order property.

Under the assumption that the interaction decays fast enough with the distance, we show that either the ground states form a one dimensional family or that there are other Birkhoff critical points which are not ground states, but lying inside the gaps left by ground states. This alternative happens if and only if a Peierls–Nabarro barrier vanishes. The main tool we use is a renormalized energy.

In the particular case that the set Λ is a one dimensional lattice and that the interaction is just nearest neighbor, our result establishes Mather's criterion for the existence of invariant circles in twist mappings in terms of the vanishing of the Peierls–Nabarro barrier.

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1 Introduction

This paper is concerned with the zero temperature behavior of some of the standard models in statistical mechanics (see, e.g., [19]). Namely, we consider a discrete set Λ occupied by particles whose state is described by a real variable. Hence, the state of the whole system is described by a configuration consisting of assigning a real variable to each site. Equivalently, a configuration is a function $u : \Lambda \to \mathbb{R}$.

We will assume that the particles at different sites interact. As in [19], this interaction is described by assigning an energy to every finite set of variables and summing over all of them. A commonly imposed restriction is that the interaction is just among nearest neighbors. In this paper, we will allow interactions which are not nearest neighbors but we will assume that they are finite range.

We will assume that the interactions are symmetric under the action of a group (say, translations or rotations) which satisfy some mild assumptions, and under the addition of an integer to all the variables. The group action induces symmetries on the set Λ as well. These symmetries are a natural feature when the physical meaning of the model is a spin variable or the position in a periodic potential.

Most importantly, we will assume that the interactions are ferromagnetic, that is the system favors particles aligning with their neighbors.

Models of this type have been considered in the Physics and Mathematics literature very often. In [8] one can have a list of models subsumed in our treatment. As a motivating prototype, we will just mention the Frenkel–Kontorova model (1) of deposition of material on a one-dimensional substrate, or of codimension 1 defects in a crystal [1, 3]. These one-dimensional models also appear naturally in mechanics as twist mappings (the functional then, has the physical interpretation of an action). See [1, 16].

The one-dimensional Frenkel–Kontorova model is described by taking $\Lambda = \mathbb{Z}$ and assigning to a configuration u, the (formal) energy

$$E(u) = \sum_{i \in \Lambda} S(u_i, u_{i+1}). \tag{1}$$

In the standard FK model, $S(x, y) = (x - y)^2 + \sin(2\pi x)$, but as remarked in the papers above, only that

$$\partial^2_{xy} S(x, y) < \delta < 0, \qquad S(x + 1, y + 1) = S(x, y) \tag{2}$$

are important for the Aubry–Mather theory.

Our goal is to consider rather general Λ, which include the standard lattices $\Lambda = \mathbb{Z}^d$ as well as Bethe lattice and other models. The main assumption will be that the set Λ will admit the action by a group G. We will also allow much more general interactions than just next neighbor interactions, but we will need to retain properties analogue to (2).

It was shown in [8] that for rather general models satisfying some rather general assumptions, detailed in Sect. 1.2, there are ground states which satisfy the order property that they do not have crossings with their translations by elements of the symmetry group (see Definition 2 below). Furthermore, it is possible to construct one of these ground states for each cocycle of the symmetry group. The order property above is one of the cornerstones of Aubry–Mather theory. We refer to [1, 16, 18] for several motivations of the order property above from solid state physics, dynamical systems and variational methods.

Since these ground states and their translations do not have crossings, we will refer to this situation as a *lamination* borrowing a name from topology. We recall that in topology a lamination is just a partition of a closed set of the space into leaves. In our case, the leaves are the graphs of the ground states, their translations and their limits. We note, however that, in our case, the leaves of the lamination may touch, which is not allowed in the most common topological definition. For this reason, we will say that our laminations are possibly singular. This singularity is related to our assumption **H9** (see Remark 24 below).

It could well happen that the set of these graphs of minimizers fills the full phase space. That is, for every site and for every value of the order parameter we can find a ground state with this value. Borrowing again a name from topology, we will refer to this situation by saying that the lamination has become a foliation (we recall that a foliation is just a lamination which covers the whole manifold we are considering and not just an strict closed subset). On the contrary, it can also happen that, for some sites, we can find an open interval so that no ground state in the family takes this value. Our goal in this paper will be to give criteria for the existence of these gaps and to obtain consequences of their existence.

The existence or not of gaps in the families of ground states has been studied in Physics (see [3] for a survey of results). When there are no gaps, it is intuitively argued that a very small external force will cause the ground state configuration to start to move since one can displace among ground states. On the other hand, when there are gaps in the family of ground states, we can expect that small (but finite) forces do not lead to motion in the configuration.

Depending on the physical interpretation of the model, the existence of gaps leads to different effects. For example in the physical interpretation of material deposited in a substratum, the absence of gaps means that the deposited material can slide-off freely, whereas the existence of gaps means that the material sticks. In the interpretation of defects, the absence of gaps means that the defects can move.

We also note that, in the case that $\Lambda = \mathbb{Z}$, $G = \mathbb{Z}$, and that the interactions are nearest neighbor, these laminations of ground states established in [8] are the celebrated minimizing Aubry–Mather sets of [1, 13]. The fact that the laminations are foliations corresponds to the Aubry–Mather sets being an invariant circle. The absence of gaps in one-dimensional cases means that the Aubry–Mather set is an invariant circle. Of course, the existence of an invariant circle is a barrier to long range transport and therefore, the question of existence of gaps has been widely studied in the dynamical systems community. Other generalizations of this theory to higher dimensions have been considered in [4–6, 12].

It was shown in [8] that, in case the cocycle is *completely irrational*, the following alternative holds: either the above lamination consists of a *foliation* made of a continuous one parameter family of ground states, or, inside any gap of the lamination, there is a well-ordered *critical point* of the energy which is not a ground state. Furthermore, these two alternatives can be ascertained by the vanishing or not of the Peierls–Nabarro barrier.

The physical meaning of these critical points is that they are metastable states.

In the case of Aubry–Mather theory, the above result becomes the celebrated criterion for existence of invariant circles for twist mappings. We note that, in the papers [13, 15], the physical interpretation of what we call in this paper energy is the action of the orbit of the twist map.

The goal of this paper is to extend such an alternative to any cocycle, *both rational or irrational*.

The assumptions we will need on the interaction are slightly more restrictive than the assumptions in [8]. Roughly, we will need that the interactions decay fast enough with the distance. The assumptions are satisfied by Frenkel–Kontorova-type models with finite range interactions. The basic model in this framework is recalled in the forthcoming Sect. 1.1.

In Sect. 1.2, we introduce the set-up for the models and in Sect. 1.4, we formulate the results of the paper. Section 2 discusses the application to more general Frenkel–Kontorova-type models.

The rest of the paper is devoted to the proofs.

We indicate that one important tool of the proof is the use of a renormalized energy. This is based on the procedure, rather customary in Physics, of using a relative energy, integrating the energy density minus the density energy of the ground state. The relative energy gives rise to a well-defined variational problem. We use methods reminiscent of the classical Ljusternik–Schnirelman theory to show that if there is a gap in the set of ground states we can construct a critical point in the gap. These critical points can be considered as metastable states. So that our result can be expressed, in somewhat informal physical terms, by saying that if the model has barriers to movement, then it has metastable states, which are also well-ordered.

We note that the proof of multiplicity results is somewhat more delicate than the proofs of existence of critical points. As it has been known for a long time, it is easy to pass to the limits of models and obtain that the limit or ground states or critical points are ground states or critical points of the limit. Nevertheless, when considering multiplicity results, it is not easy to show that the limits are different. Hence, when considering multiplicity results, we cannot use cut-offs or approximating the models in any other way. In particular, it is not immediate that one can use the results for strictly positive twist to systems where the twist may vanish. From the physical point of view, this is related to the fact that mobility of the ground states is a rather subtle phenomenon.

1.1 A Particular Case: the Standard Frenkel–Kontorova model

Though the framework we will deal with is quite general, some of the results may be better visualized for the standard, one-dimensional, Frenkel–Kontorova model. The Hamiltonian of this model may be taken as

$$\sum_{i\in\mathbb{Z}} |u_i - u_{i+1}|^2 + V(u_i), \tag{3}$$

where V is a smooth, 1-periodic, potential (see [4, 9] and Sect. 2 here below).

The advantage of thinking to the model above as a paradigmatic example is that the "Euclidean", low-dimensional structure of the group $\mathbb{Z}$ makes the symmetries easy to visualize, and the algebraic notion of cocycle reduces to the one of linear functions.

In [4, 9], more general models than (3) were studied, and ground states at bounded distance from any assigned linear function were constructed. Also, it is proven that these ground states do not cross under integer translations, that is, if u is such a ground state, given $\ell, s \in \mathbb{Z}$, one of the following three alternative holds:

- $u_{i+\ell} + s > u_i$,
- $u_{i+\ell} + s < u_i$,
- $u_{i+\ell} + s = u_i$,

for all $i \in \mathbb{Z}$. The point of the above alternative is that the same comparison happens for all the i. For each i, of course, we have one comparison, but, for general configurations, this comparison would depend on i. The configurations that satisfy this properties are very special.

This implies that the well-ordered ground states—together with their translations and addition of integers—give rise to a lamination. It is then shown in [9] that the subsequent dichotomy holds: either such a lamination is made of a *one-parameter continuum* of ground states, or there are gaps and a *Birkhoff critical point of the interaction inside each gap*, which is not a ground state.

The above results have been extended to very general interactions and group symmetries in [8]. There, the existence of ground state laminations was proven for any prescribed cocycle, and the dichotomy was proven for completely irrational cocycles (that is, cocycles having "irrational slope in any direction").

We note that in the one-dimensional case, it is known that when the rotation number is irrational and the set of ground states is not a foliation, there are uncountably many critical points in the gaps which are, nevertheless, not well-ordered [2, 10, 14]. In this paper, we have not considered the existence of these critical points which are not well-ordered.

The purpose of this paper is then, under appropriate further assumptions, to extend the dichotomy to partly rational cocycles (see Theorem 7 below).

For this, we need a first set of assumptions (namely, **G1–G4** below), dealing with the group structure of the symmetries. Conditions **H1–H10** will then fix the required properties of the Hamiltonian (such as symmetry, coercivity, ferromagnetism and range of interaction).

Assumptions **A1–A4** will be technical conditions to bound the combinatorics of the resonances and the interactions through the boundaries of finite domains. They can be considered as an strengthening of the decay properties of the Hamiltonian and are satisfied if the interactions are finite range or if they decay fast enough (as expected, there should be some relation between the speed of decrease of the interaction and the rate of growth of volumes of the ball in the crystal).

1.2 Description of Models and Assumptions

We now formally introduce our set-up. The set-up is rather general. This allows to cover at the same time several models that have been considered in the literature (see [8]). Moreover, this generality allows to emphasize what are the essential ingredients in the arguments, which are mainly symmetry, ferromagnetism (as well as some very minor regularity). The arguments do not become significantly simpler by making them in the particular case of lattices. Also, we note that the results presented here are, to the best of our knowledge, new even in the case of lattices.

We consider a discrete countable set Λ, which admits an action by a group G. We think that each of the elements in Λ is occupied by a system whose state is determined by a real variable. Hence, the configuration of the system are given by assignments of real variables to each site. We will denote a configuration by $u : \Lambda \to \mathbb{R}$.

We will also assume that there is an interaction given by a collection of potentials H_B, associated to any finite subset B of Λ.

From the statistical mechanics viewpoint, Λ may be thought as a crystal, which possesses the symmetries induced by G, and each H_B describes the interaction between the particles in the set B.

The most customary example in statistical mechanics is when $\Lambda := \mathbb{Z}^d =: G$ and the action of G on Λ is given by translations. The present framework applies to a variety of models (see [8] for a presentation of different models). In particular, in crystallography, the symmetry groups of crystals, often include rotations or reflections. Nevertheless, the reader may want to keep this example in mind. A detailed verification for models in $\Lambda := \mathbb{Z}^d =: G$ will be done in Sect. 2.

As customary in statistical mechanics (see, e.g., [19]), the interaction potential is given by the formal sum

$$S(u) = \sum_{B\subset\Lambda} H_B(u), \tag{4}$$

where $u:\Lambda\to\mathbb{R}$ is called a *configuration* on Λ.

We will be interested in equilibria and in and in particular in ground states. A fuller discussion of these concepts will be done in Sect. 1.3.

Now, we start to formulate the assumptions and to recall some well-known notions.

We write G in multiplicative notation, that is the group operation between $g_1, g_2 \in G$ will be denoted by $g_1 g_2$. The identity element is denoted by Id, namely $g\,\mathrm{Id} = \mathrm{Id}\,g = g$ for any $g \in G$.

We recall that a cocycle on G is a map

$$\sigma : G \to \mathbb{R}$$

such that

$$\sigma(\gamma\gamma') = \sigma(\gamma) + \sigma(\gamma'). \tag{5}$$

Since G acts on the countable set Λ, given $p \in \Lambda$ and $\gamma \in G$, we denote the action of γ on p by γp.

1.2.1 Assumptions on the Action of the Group G

We now list the hypotheses that we assume on the group G. They are the same as those in [8].

G1: The group G is *finitely generated*.
G2: There is a *finite fundamental domain* for the action, i.e., a finite subset F of Λ which intersects each orbit of G in exactly one point.
G3: The group G acts on Λ *without nontrivial stabilizers*, i.e.: if $\gamma \in G$ is such that there exists $p \in \Lambda$ in such a way that $\gamma p = p$, then $\gamma = \mathrm{Id}$.
G4: The group G is *residually finite*, i.e., for each element of G other than the identity, there is a normal, finite index subgroup of G which does not contain it.

Note that these assumptions on G are very mild (see also [5, 8] for a discussion on this point).

Up to identifying a representation class of Λ/G with the corresponding element of F, one can identify Λ/G with F.

Also, by **G1**, G is generated by a finite set of elements, say $\{g_1^\star, \dots, g_d^\star\}$. Then, given a cocycle σ, we define its norm by

$$|||\sigma||| := \sup_{1\le k\le d} |\sigma(g_k^\star)|. \tag{6}$$

Moreover, any cocycle σ defines a configuration u_σ on Λ as follows. Since F is the fundamental domain, for any $p \in \Lambda$ there exists a unique $q \in F$ so that $p = \gamma q$, for some $\gamma \in G$. Then, we define

$$u_\sigma(p) = \sigma(\gamma). \tag{7}$$

Note that the above γ is unique, thanks to **G3**, and so (7) is a good definition.

Let us emphasize that we are not assuming that the group G is the maximal symmetry group acting on the lattice. The system may have a larger symmetry group. Nevertheless, sometimes one obtains more cocycles by considering a smaller group.

Assumption **G3** is not very crucial for the argument developed here. The argument could work similarly with a finite stabilizer, but the argument would become somewhat more cumbersome to write. If the system has point symmetries, we just note that, by the above remark, we can ignore them. It is also possible to introduce several copies of the system and make the group G move along the different copies.

1.2.2 Assumptions on the Interaction Potential H

Let us now formalize the concept of interaction potential in (4).

Given $K \geq 0$ and a cocycle σ, we define $\mathcal{O}_\sigma^K$ to be the space of all configurations u such that

$$\sup_{p\in\Lambda} |u(p) - u_\sigma(p)| \leq K. \tag{8}$$

Let $\mathcal{S}$ denote a collection of nonempty, finite subsets of Λ. Then, we consider the collection of maps

$$H = \{H_B, B \in \mathcal{S}\},$$

where each H_B is a real valued function from the configurations u of Λ.

We will be making the following assumptions on the potential, which are the same as those in [8]. Later, we will include some other assumptions on the potential which are used in this paper.

H1: If

$$u(p) = v(p) \quad \text{for any } p \in B$$

then

$$H_B(u) = H_B(v).$$

H2: Fixed $K \geq 0$, for any finite subset X of Λ and any configuration $u \in \mathcal{O}_\sigma^K$, the series

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u) \tag{9}$$

converges uniformly.

By **H1**, we may regard any function H_B as being a function from $\mathbb{R}^B$ to $\mathbb{R}$. We sometimes write $u_p := u(p)$, for $p \in \Lambda$, and take derivatives of H_B with respect to u_p. Also, by possibly neglecting the sets B in $\mathcal{S}$ for which $H_B \equiv 0$, without loss of generality we will suppose that $H_B \not\equiv 0$ if $B \in \mathcal{S}$.

The physical meaning of assumption **H2** is that, if we fix configurations in the class $\mathcal{O}_\sigma^K$, we can talk about the energy of a finite set. Later, in **H8** we will assume that the forces (derivatives of the energy) acting on a particle make sense and also that the derivatives of these one particle forces with respect to the values of other particles make sense, so that the convergence of the series in (9) is not only in the values but also in the values of the other derivatives.

The action of G on Λ extends to the space of configurations. That is, if $\gamma \in G$, we define

$$\mathcal{T}_\gamma u(p) := u(\gamma p),$$

for any $p \in \Lambda$. Given $\ell \in \mathbb{R}$, we also set

$$\mathcal{R}_\ell u := u + \ell.$$

Thus, we assume:

H3: The potential $H = \{H_B\}$ is G-*invariant*, i.e.: if $B \in \mathcal{S}$, then $\gamma B \in \mathcal{S}$ and

$$H_{\gamma B}(u) = H_B(\mathcal{T}_\gamma u)$$

for any $\gamma \in G$ and any configuration u on Λ.

H4: H has *periodic phase*, i.e.,

$$H_B(\mathcal{R}_s u) = H_B(u)$$

for any configuration u, any $B \in \mathcal{S}$ and any $s \in \mathbb{Z}$.

H5: The action of the group on the potential is *nontrivial*, i.e., we suppose that if $g \in G$ is such that $gB = B$ for some $B \in \mathcal{S}$, then g is the identity.

H6: H is *coercive*, i.e., we suppose that $H_B(u) \geq 0$ for any $B \in \mathcal{S}$ and

$$H_F(u) \geq \kappa \max_{i,j \in F} |u_i - u_j|^\theta - \chi,$$

for some κ, χ, $\theta > 0$.

H7: There exist ϑ, $S > 0$ and $\varsigma : \mathcal{S} \to [0, +\infty)$ in such a way that

$$H_B(\sigma) \leq \varsigma(B)[\|\!|\sigma|\!\|^\vartheta + 1],$$

for any cocycle σ and any $B \in \mathcal{S}$, and

$$\sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset}} \varsigma(B) \leq S \sharp X,$$

for any $X \subset \Lambda$.

Given κ, χ, S, θ, ϑ as in **H6** and **H7**, we set

$$K^\star_\sigma := \begin{cases} 9 & \text{if } \sharp(\Lambda/G) = 1, \\ 9 + \left(\frac{S(\|\!|\sigma|\!\|^\vartheta + 1)\sharp F + \chi}{\kappa}\right)^{1/\theta} & \text{if } \sharp(\Lambda/G) \geq 2. \end{cases}$$

We also set $\mathcal{O}_\sigma := \mathcal{O}_\sigma^{K^\star_\sigma}$.

Additionally, we assume that

H8: For any $K \geq 0$ and any cocycle σ, the interaction potential H is C^2-bounded on $\mathcal{O}_\sigma^K$, that is, for any $B \in \mathcal{S}$, the map H_B is twice differentiable and

$$\sup_{u \in \mathcal{O}_\sigma^K} \sup_{p \in \Lambda} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \left| \frac{\partial H_B}{\partial u_p}(u) \right| + \sup_{u \in \mathcal{O}_\sigma^K} \sup_{p \in \Lambda} \sum_{\substack{B \in \mathcal{S} \\ B \ni p}} \sum_{q \in B} \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(u) \right| < +\infty. \tag{10}$$

H9: H satisfies the *ferromagnetic* (or *twist*) condition, i.e.

$$\frac{\partial^2 H_B}{\partial u_p \partial u_q}(u) \leq 0,$$

for any $u \in \mathcal{O}_\sigma^K$, any $K \geq 0$ and any p, q in Λ, with $p \neq q$.

H10: There exist $\delta : \mathcal{S} \to [0, +\infty)$ in such a way that

- $\delta(B) = \delta(gB)$, for any $B \in \mathcal{S}$ and any $g \in G$.
- For any $K \geq 0$ and any finite set $X \subset \Lambda$,

$$\lim_{R\to+\infty} \sup_{u\in\mathcal{O}_\sigma^K} \sum_{\substack{B\in\mathcal{S} \\ B\cap X\neq\emptyset \\ \delta(B)>R}} H_B(u) = 0.$$

- Given any $R > 0$ and any finite set $X \subset \Lambda$, there exists a finite set $Y \subset \Lambda$ in such a way that

$$\bigcup_{\substack{B\in\mathcal{S} \\ B\cap X\neq\emptyset \\ \delta(B)\leq R}} B \subset Y.$$

In concrete applications, one may take δ to be diameter of the set B (see, for instance, [8] and Sect. 2 here below).

The meaning of assumption **H9** is that the configurations can lower their energy by aligning themselves with other sites, so it is an assumption on ferromagnetism. We note that in the cases of the one-dimensional lattices in [13, 15], it corresponds to the twist condition. We note that we are allowing non-strict inequalities, while, in the applications to dynamical systems, it is customary to assume the stronger hypotheses that the interaction is just to nearest neighbors and that $\partial_{u_i}\partial_{u_{i+1}} H_{\{i,i+1\}} \leq -a$ for some $a > 0$.

1.2.3 Partial Order Among Configurations

Note that the space of configurations is endowed with a natural *partial ordering*:

Definition 1 We say that $u \leq v$ if $u(p) \leq v(p)$ for any p in Λ.

Then, generalizing a standard terminology in Aubry–Mather theory, we introduce the following order property:

Definition 2 Given a cocycle σ of G, we say that a configuration u satisfies the **Birkhoff property** for σ if

$$\lfloor \sigma(\gamma) \rfloor \leq u(\gamma p) - u(p) \leq \lceil \sigma(\gamma) \rceil \tag{11}$$

for every $p \in \Lambda$ and $\gamma \in G$.

In (11), we made use of the standard notation for which $\lfloor r \rfloor$ (resp., $\lceil r \rceil$) denotes the largest integer less than or equal to r (resp., the smallest integer not less than r).

We remark that u satisfies the Birkhoff property if and only if the following holds: if $\gamma \in G$ and $s \in \mathbb{Z}$ are such that $\sigma(\gamma) \geq s$ (resp., $\sigma(\gamma) \leq s$), then $\mathcal{T}_\gamma u \geq \mathcal{R}_s u$ (resp., $\mathcal{T}_\gamma u \leq \mathcal{R}_s u$).

The set of Birkhoff configurations in $\mathcal{O}_\sigma$ will be denoted by $\mathcal{B}_\sigma$. Note that $u_\sigma(\gamma p) - u_\sigma(p) = \sigma(\gamma)$, by (5) and (7), thence u_σ belongs to $\mathcal{B}_\sigma$, which is then non-empty.

Definition 3 We say that u is a **ground state** (or, in the terminology of [17], a **class-A minimizer**) if

$$\sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(u) \le \sum_{\substack{B\in\mathcal{S}\\ B\cap X\neq\emptyset}} H_B(v)$$

for any finite subset X of Λ and any configuration v such that $u = v$ on $\Lambda - X$.

Definition 4 We say that a collection $\mathcal{L}$ of ground states is a (possibly singular) lamination when

- (i) $u, v \in \mathcal{L}$ implies that we have $u \ge v$ or $u \le v$.
- (ii) The set $\mathcal{L}$ is closed under pointwise limits.

When (i) is replaced by

- (i′) $u, v \in \mathcal{L}$ implies that we have $u \equiv v$, or $u_i < v_i$ for all $i \in \Lambda$, or $u_i > v_i$ for all $i \in \Lambda$,

we will say that $\mathcal{L}$ is a lamination.

When $\mathcal{L}$ satisfies, in addition to the above properties, that

- (iii) For every $i \in \Lambda$, $\{u_i \mid u \in \mathcal{L}\} = \mathbb{R}$

we will say that $\mathcal{L}$ is a foliation.

Note that the Birkhoff condition is closed under pointwise limits. Note also that if a configuration is Birkhoff, all its translations and addition of integers are also Birkhoff. Hence, the closure of the translations and additions of integers of a Birkhoff configuration is a (possibly singular) lamination.

1.3 Ground States Versus Critical Points

Let us now introduce the equilibrium solutions we are interested in:

Definition 5 We say that a configuration u is a **solution of our variational equation** (or a **critical point**) if, for any $p \in \Lambda$,

$$\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u) = 0. \tag{12}$$

Note that the series in (12) is absolutely convergent, thanks to **H8**.

The following result has been proved in Theorem 1 of [8]:

Theorem 6 *Assume* **G1–G4** *and* **H1–H10**. *Then, given any cocycle σ, there exists a ground state u of the interaction S such that*

$$|u(p) - u_\sigma(p)| < K_\sigma^\star. \tag{13}$$

Moreover, $u \in \mathcal{B}_\sigma$.

These ground states induce, by the natural projection, a possibly singular lamination in $F \times \mathbb{T}^1$, that is, the (closure of the) graphs of such ground states are organized into leaves that do not cross.

Note that the graphs of ground states and the translations via $\mathcal{T}_\gamma$ and $\mathcal{R}_\ell$ do not cross. We also note that the pointwise limit of ground states is a ground state too (see, e.g., the easy argument in [8]).

Theorem 6 may be better visualized in the easier case of the standard Frenkel–Kontorova model: in this setting, (13) reduces to the fact that u has bounded distance from a linear function (in fact, in such a case, this distance is bounded uniformly with respect to the slope, see [4, 9]). Also, in the standard Frenkel–Kontorova model, the lamination of Theorem 6 is obtained by (taking the closure of) the translations $u_{i+\ell} + s$, for integer ℓ and s.

We now define

$$\mathcal{H} := \{g \in G \mid \sigma(g) \in \mathbb{Z}\}.$$

In case $\mathcal{H}$ contains only the identity (that is, if the cocycle σ is "completely irrational"), it has been proved in Theorem 2 of [8] that it is possible to find other critical points (which are not minimal) inside a gap of the lamination of ground states given by Theorem 6 here above. The situation in which $\mathcal{H}$ contains only the identity may be seen as the "typical" case of the irrational cocycles (for instance, when cocycles are just linear functions and $G = \mathbb{Z}^d$, it reduces to prescribe rationally independent slopes).

The purpose of this paper is to construct critical points inside the gaps of the ground state lamination of Theorem 6 under more relaxed conditions, which take into account "rationally dependent slopes" too. This goal will be accomplished in Theorem 7 below, which will be the main result of the present paper.

The price to pay for such generalization is that we need to make *assumptions on the range of the interaction potential*. Roughly speaking, we will want to avoid that the potential range "wraps around" the fundamental domain too much. Thus, the models we are dealing with may be seen as a generalization of finite range interaction systems.

The assumptions we now introduce may in fact look quite abstract, but they are also very general: as an example, we will show later in Sect. 2 that multidimensional Frenkel–Kontorova models fulfill such conditions.

In further detail, the auxiliary assumptions we are going to take in this paper are the following ones. We suppose that there exists a sequence of normal, finite index subgroups $H_n \subset H_0 := H \subset \mathcal{H}$ in such a way that $(\Lambda/H_n) \subset (\Lambda/H_{n+1})$ which satisfy[1] the following conditions:

A1: Given any finite set $X \subset \Lambda$, there exists $n \in \mathbb{N}$ in such a way that $X \subset \Lambda/H_n$.

A2: There exists $k^\star \in \mathbb{N}$ such that if $B \in \mathcal{S}$ and $n \in \mathbb{N}$ are such that $n \geq k^\star$ and $B \cap (\Lambda/H_n) \neq \emptyset$, then either $B \subset \Lambda/H_n$ or

$$B \subset (\Lambda/H_{n+k^\star}) - (\Lambda/H_{n-k^\star}).$$

Of course, no confusion should arise between the subgroups H_n and the Hamiltonian H_B, since the context should clarify any ambiguity.

Condition **A1** just says that the fundamental domains Λ/H_n exhaust the whole lattice Λ as n increases (and thus it makes **G4** more precise). We also remark that, since $\Lambda/H =$

[1] As already mentioned in Sect. 1.2.1, we are identifying, here and in the rest of the paper, the quotient Λ/H with a suitable fundamental domain for the action of the group H on the lattice Λ. Hence, the notation $\Lambda/H' \subset \Lambda/H''$ has to be read in the geometry of the corresponding fundamental domains, and it just means that there exist a suitable fundamental domain F' (resp., F'') for the action of the group H' (resp., H'') on Λ, in such a way that $F' \subset F''$.

$\Lambda/H_0 \subset \Lambda/H_{k^\star}$, it follows from **A2** that if B intersects Λ/H, then $B \subset \Lambda/H_{2k^\star}$ (that is, roughly speaking, the sets in $\mathcal{S}$ "cannot be too large", at least "in the direction of H").

We now introduce an equivalence relation on subsets of Λ, induced by the subgroup H. Given $A, B \subset \Lambda$, we say that $A \sim_H B$ if there exists $h \in H$ in such a way that $A = hB$. The equivalence class of a set B with respect to this relation will be denoted by $[B]_H$.

With this notation, we assume that:

A3: If $A, B \in \mathcal{S}$ are such that $A \sim_H B$ and $A \cap B \neq \emptyset$, then $A = B$.

A4: For any $n \in \mathbb{N}$, $n \geq k^\star$, there exist $c(n) \geq 0$ such that

$$\sum_{p \in (\Lambda/H_{n+k^\star}) - (\Lambda/H_{n-k^\star})} u_p = c(n) \sum_{p \in \Lambda/H} u_p, \tag{14}$$

for any configuration u such that $u_p = u_{hp}$ for any $p \in \Lambda$ and any $h \in H$.

Furthermore, the following limit holds:

$$\lim_{n \to +\infty} \frac{c(n)}{\sharp(H/H_n)} = 0, \qquad \lim_{n \to +\infty} \sup_{B \in \mathcal{S}} \left| \frac{\sharp\{A \in [B]_H \text{ s.t. } A \subset \Lambda/H_n\}}{\sharp(H/H_n)} - 1 \right| = 0. \tag{15}$$

Roughly speaking, condition **A3** states that the sets in $\mathcal{S}$ "cannot overlap" under the periodicity induced by H and, again, it may be seen as a requirement on the size of these sets (in the sense that a set does not overlap with respect to a periodicity notion as long as "its diameter is less than the period").

The reason why we introduce $H = H_0$ (instead of working directly with $\mathcal{H}$) is explained by condition **A3**: there we suppose that H is so "small" that the interaction sets do not overlap with respect to the H-symmetry (if the interactions are "long range", $\mathcal{H}$-symmetries may provide overlaps).

Condition **A4** is more technical and it is needed to control the combinatorics of different periodic identifications. Such condition may be weakened: for instance, in the second limit of (15), 1 could be replaced by any $\ell > 0$ (see the argument ending the proof of Lemma 17).

We will show that all the above conditions are satisfied, for instance, in the case of finite range interactions on the lattice $\mathbb{Z}^d$, such as multidimensional Frenkel–Kontorova models (see Sect. 2 for details).

1.4 Statement of Results

Thus, the main result of this paper is:

Theorem 7 *Assume* **G1–G4**, **H1–H10** *and* **A1–A4**. *Let* $u^{(0)}, u^{(1)} \in \mathcal{B}_\sigma$. *Assume that the graphs of* $u^{(0)}$ *and* $u^{(1)}$ *belong to the (possibly singular) lamination of ground states induced by Theorem 6. Suppose that* $u^{(1)} \geq u^{(0)}$ *and* $u^{(1)} \not\equiv u^{(0)}$. *Then, there exists* $u \in \mathcal{B}_\sigma - \{u^{(0)}, u^{(1)}\}$ *such that*

$$u^{(1)} \geq u \geq u^{(0)}$$

and which is a solution of the variational equation.

We remark that there are always more than one ground state in $\mathcal{B}_\sigma$, because if u is one of these ground states, so is $\mathcal{T}_\gamma \mathcal{R}_\ell u$ (recall Theorem 6), thence the assertion of Theorem 7 is non-empty.

Note that the interesting case of the result is when the $u^{(0)}, u^{(1)}$ are at the boundary of a gap in the lamination: then, the solution thus produced is not included in the lamination (otherwise, we can take as new critical point some other leaf in the lamination between $u^{(0)}$ and $u^{(1)}$).

It could happen that the new critical point is also a ground state. In such a case, we can add the new ground state, its translations and limits to the lamination of ground states and we can start the process again.

Hence, we conclude that either there is a lamination by ground states that covers all the fundamental domain or there is a critical point inside the gaps which is not a ground state.

Theorem 7 is a generalization of the results of [15], were it is shown that, for the models of twist maps considered in Aubry–Mather theory, either there is an invariant circle (the lamination covers the fundamental domain) or there are equilibrium points which are not ground states. We will also see that there is a natural extension of the Peierls–Nabarro barrier considered in [15] to the models that we consider in this paper.

The proof of Theorem 7 we will present here is based on the study a gradient flow. Similar arguments have appeared in [2, 5, 7, 8]. In particular, the technique developed in [8] will be extended to overcome the additional complication of the possible resonances given by the case of nontrivial $\mathcal{H}$. This will be obtained by a careful study of the group identifications naturally induced by $\mathcal{H}$.

2 Frenkel–Kontorova Models with Finite Range Potentials

As an example to which Theorem 7 applies, we consider the periodic finite range interaction potentials. Namely, we observe that:

Proposition 8 *Let $G = \Lambda := \mathbb{Z}^d$, with the sum as group action. In this case, the cocycles are linear functions from $\mathbb{Z}^d$ to real, which are given by specifying the d values of the action on the generators.*

Suppose that there exists $R \geq 0$ such that any $B \in \mathcal{S}$ has diameter less than R. Then, conditions **A1**–**A4** *are fulfilled.*

Proof Given a cocycle σ we write

$$\sigma(i) = \omega \cdot i,$$

for any $i \in \mathbb{Z}^d$.

Then,

$$\mathcal{H} = \{i \in \mathbb{Z}^d \mid \omega \cdot i \in \mathbb{Z}\},$$

and its span over $\mathbb{R}$, which will be denoted by $\langle \mathcal{H} \rangle$, is a linear space and so it is generated by suitable linearly independent integer vectors $e_1, \dots, e_k \in \mathbb{Z}^d - \{0\}$ for some $k \geq 0$ (which is the "number of resonances" of ω). In fact, we will suppose $k \geq 1$, otherwise **A1**–**A4** are easily checked by choosing $H_n := H = \{0\}$.

We complete $\mathcal{H}$ into $\mathbb{R}^d$, namely, we consider a maximal $(n-k)$-dimensional linear lattice $\widetilde{\mathcal{H}} \subset \mathbb{Z}^d$ such that $\mathbb{R}^d = \langle \mathcal{H} \rangle \oplus \langle \widetilde{\mathcal{H}} \rangle$.

Given any

$$a = \sum_{j=1}^{k} a_j e_j + \alpha \in \mathbb{R}^d$$

with $a_j \in \mathbb{R}$ and $\alpha \in \langle \widetilde{\mathcal{H}} \rangle$, we define

$$|a|_* := \sqrt{\sum_{j=1}^{k} |a_j|^2 + |\alpha|^2}.$$

It follows that $|\cdot|_*$ induces a norm on $\mathbb{R}^d$, which is therefore equivalent to the Euclidean norm $|\cdot|$. In particular, there exists $\lambda \in (0,1]$ in such a way that

$$|a| \geq \lambda \max\{|e_1|, \dots, |e_k|\}|a|_* \geq \lambda |a_j||e_j|, \quad \text{for any } j = 1, \dots, k. \tag{16}$$

We write

$$\mathcal{H} = \left\{ \sum_{j=1}^{k} m_j e_j \,\middle|\, m_j \in \mathbb{Z} \right\}$$

and we take $\nu \in 2\mathbb{N}$ such that $(R+1)/\lambda < \nu$. We set

$$H := \left\{ \sum_{j=1}^{k} \nu m_j e_j \,\middle|\, m_j \in \mathbb{Z} \right\}$$

and, for $n \geq 1$,

$$H_n := \left\{ \sum_{j=1}^{k} \nu m_j e_j \,\middle|\, m_j \in 2n\mathbb{Z} \right\}.$$

Then, we may choose as fundamental domains

$$\Lambda/H = \left\{ \sum_{j=1}^{k} m_j e_j + \beta \,\middle|\, m_j \in \mathbb{Z} \cap [-\nu/2, \nu/2),\ \beta \in \widetilde{\mathcal{H}} \right\},$$

$$\Lambda/H_n = \left\{ \sum_{j=1}^{k} m_j e_j + \beta \,\middle|\, m_j \in \mathbb{Z} \cap [-n\nu, n\nu),\ \beta \in \widetilde{\mathcal{H}} \right\}.$$

With this notation, the fact that $(\Lambda/H_n) \subset (\Lambda/H_{n+1})$ and condition **A1** are obvious.

We now prove **A2**. For this, we take $k^\star$ to be the smallest natural number larger than $(R/\lambda) + 2$. Let now suppose that $B \in \mathcal{S}$ is such that $B \cap (\Lambda/H_n) \neq \emptyset$ and $B \not\subset \Lambda/H_n$, with $n \geq k^\star$. Then, there exist two points

$$b^{(i)} = \sum_{j=1}^{k} b_j^{(i)} e_j + \beta^{(i)} \in B, \quad b_j^{(i)} \in \mathbb{R},\ \beta^{(i)} \in \langle \widetilde{\mathcal{H}} \rangle,\ i = 1, 2,$$

in such a way that $|b_j^{(1)}| \leq n\nu$ for any $1 \leq j \leq k$ and $|b_{j_0}^{(2)}| \geq n\nu$ for some $1 \leq j_0 \leq k$. Then, given any point $b \in B$, we write

$$b = \sum_{j=1}^{k} b_j e_j + \beta, \quad b_j \in \mathbb{R},\ \beta \in \langle \widetilde{\mathcal{H}} \rangle,$$

and we deduce from (16) that

$$R \geq \left|b - b^{(i)}\right| \geq \lambda \left|b_j - b_j^{(i)}\right|$$

for any $j = 1, \dots, k$ and $i = 1, 2$. Therefore,

$$|b_j| \leq (R/\lambda) + \left|b_j^{(1)}\right| \leq (R/\lambda) + n\nu < (n + k^\star)\nu$$

and

$$|b_{j_0}| \geq \left|b_{j_0}^{(2)}\right| - \left|b_{j_0} - b_{j_0}^{(2)}\right| \geq n\nu - (R/\lambda) > (n - k^\star)\nu.$$

This shows that $B \subset (\Lambda/H_{n+k^\star}) - (\Lambda/H_{n-k^\star})$ and proves **A2**.

We now check condition **A3**. For this, suppose that $q \in A \cap B$, with $A = B + h$, for some $h \in H$. Let

$$h = \sum_{j=1}^{k} \nu h_j e_j$$

with $h_j \in \mathbb{Z}$.

Since $q + h \in B + h = A$, we have that both $q + h$ and q belong to A and so, from (16),

$$R \geq |(q + h) - q| = |h| \geq \lambda\nu |h_j|$$

for any $j = 1, \dots, k$. If one of the h_j were not zero, we would then have that $R \geq \lambda\nu > R$, which is a contradiction. Therefore, $h_j = 0$ for any $j = 1, \dots, k$ and so $h = 0$. This shows that $A = B$ and so proves that **A3** is fulfilled.

Let us now prove **A4**. We notice that

$$\sharp(H/H_n) = \sharp(\mathbb{Z}/(2n\mathbb{Z}))^k = (2n)^k. \tag{17}$$

On the other hand, given $n > m$, we observe that $p \in (\Lambda/H_n) - (\Lambda/H_m)$ if and only if

$$\mathbb{Z}^d \ni p = \sum_{j=1}^{k} p_j e_j + \beta$$

with $\beta \in \langle\widetilde{\mathcal{H}}\rangle$, $p_j \in \mathbb{Z} \cap [-n\nu, n\nu)$ for any $j = 1, \dots, k$, and $p_{j_0} \notin [-m\nu, m\nu)$ for some j_0.

Note also that, if u is as requested in **A4**, we have that

$$u(p) = u\left(\sum_{j=1}^{k} p_j e_j + \beta\right) = u\left(\sum_{j=1}^{k} [p_j]_\nu e_j + \beta\right),$$

where $[r]_\nu$ denotes the equivalence class of $r \in \mathbb{Z}$ with representatives, say, the integers in $[-\nu/2, \nu/2)$.

So, (14) holds, with

$$c(n) = \sharp\{p_1, \dots, p_k \in \mathbb{Z} \cap [-(n + k^\star)\nu, (n + k^\star)\nu) \mid$$
$$p_{j_0} \notin [-(n - k^\star)\nu(n - k^\star)\nu) \text{ for some } j_0\}.$$

Consequently,

$$\begin{aligned}
c(n) &\le \sharp\{p_1,\dots,p_k\in\mathbb{Z} \mid |p_j|\le (n+k^\star)\nu \text{ for any } j \\
&\qquad \text{and } p_{j_0}\notin[-(n-k^\star)\nu,(n-k^\star)\nu) \text{ for some } j_0\} \\
&\le \sum_{j_0=1}^{k} \sharp\{p_1,\dots,p_k\in\mathbb{Z} \mid |p_j|\le (n+k^\star)\nu \text{ for any } j \\
&\qquad \text{and } p_{j_0}\notin[-(n-k^\star)\nu,(n-k^\star)\nu)\} \\
&\le k\cdot\sharp\{p_1,\dots,p_k\in\mathbb{Z} \mid |p_1|,\dots,|p_{k-1}|\le (n+k^\star)\nu \\
&\qquad \text{and } (n-k^\star)\nu\le|p_k|\le(n+k^\star)\nu\} \\
&\le \nu^k k\cdot(2(n+k^\star)+1)^{k-1}\cdot(4k^\star+1) \\
&\le 2^{k+3}\nu^k k k^\star (n+k^\star+1)^{k-1}.
\end{aligned}$$

From this and (17), the first limit in (15) follows.

We now take care of the second limit in (15). For this, we consider the map

$$\begin{aligned}
P : \Lambda/H_n &\longrightarrow ([-n\nu,n\nu)\cap\mathbb{Z})^k, \\
\sum_{j=1}^{k} m_j e_j+\beta &\longmapsto (m_1,\dots,m_k)
\end{aligned}$$

and we observe that

$$\text{if } A\in[B]_H,\quad \text{then } P(A)=P(B)+\nu h, \tag{18}$$

for some $h\in\mathbb{Z}^k$.

Since the diameter of B is less than R, if $p=(p_1,\dots,p_k), q=(q_1,\dots,q_k)\in P(B)$, we deduce from (16) that

$$k^\star > \frac{R}{\lambda} \ge |p_i-q_i| \quad \text{for } i=1,\dots,k. \tag{19}$$

Let now $n\ge k^\star$. Given any $B\in\mathcal{S}$, fix $p\in P(B)$. Then, there exist $2(n-k^\star)$ consecutive integers $j\in\mathbb{Z}$ such that

$$p+\nu(j,0,\dots,0)\in[-(n+k^\star)\nu,(n-k^\star)\nu)\times\mathbb{Z}^{k-1}.$$

Thus, from (19),

$$P(B) + \nu(j, 0, \dots, 0) \subset [-n\nu, n\nu) \times \mathbb{Z}^{k-1}.$$

By repeating the above argument to any of the k coordinates of p, we get that there exist $(2(n - k^\star))^k$ integer vectors $J \in \mathbb{Z}^k$ in such a way that

$$P(B) + \nu J \subset [-n\nu, n\nu)^k.$$

Thus, recalling (18),

$$\begin{aligned}
&\sharp\{A \in [B]_H \text{ s.t. } A \subset \Lambda/H_n\} \\
&\quad = \sharp\{J \in \mathbb{Z}^k \text{ s.t. } P(B) + \nu J \subset [-\nu n, \nu n)^k\} \\
&\quad \geq (2(n - k^\star))^k.
\end{aligned} \tag{20}$$

On the other hand, obviously,

$$\sharp\{J \in \mathbb{Z}^k \text{ s.t. } P(B) + \nu J \subset [-n\nu, n\nu)^k\} \leq \sharp(\mathbb{Z}^k \cap [-n, n]^k) \leq (2n + 1)^k.$$

The latter estimate, (20) and (17) yield the second limit in (15), thus checking condition **A4**. □

It easily follows from Proposition 8 that Theorem 7 applies to multidimensional Frenkel–Kontorova-type models, as next result states. We note that in [9] one can find a direct verification for some generalized Frenkel–Kontorova models that include also the possibility of interaction being by hard springs.

In particular, we consider here the system

$$\sum_{\substack{i,j \in \mathbb{Z}^d \\ |i-j| \leq R}} |u_i - u_j - a|^2 + \sum_{i \in \mathbb{Z}^d} V(u_i), \tag{21}$$

for given $a \in \mathbb{R}$ and $R \geq 0$.

The system in (21) generalizes the one-dimensional Frenkel–Kontorova model with zero equilibrium distance in the absence of external fields.

Corollary 9 *Let $V \in C^\infty(\mathbb{R})$ be such that*

$$V(r + 1) = V(r) \tag{22}$$

for any $r \in \mathbb{R}$. Then, the results of Theorems 6 and 7 apply to the system in (21), by choosing $G = \Lambda := \mathbb{Z}^d$, with the group action of G on Λ being the sum, and the linear functions from $\mathbb{R}^d$ to $\mathbb{R}$ as the set of cocycles.

Proof By possibly replacing $V(r)$ with $V(r) - \min V$, we may suppose that

$$V(r) \geq 0 \quad \text{for any } r \in \mathbb{R}. \tag{23}$$

Conditions **G1–G4** are obviously fulfilled here. Let $\mathcal{S}$ be the collection of all the nonempty subsets of $\mathbb{Z}^d$ with diameter less than or equal to R containing no more than two points. That is, given $B \subset \mathbb{Z}^d$, we have that $B \in \mathcal{S}$ if and only if either B is a point or $B = \{i, j\}$ with $|i - j| \leq R$.

We define

$$H_B(u) := \begin{cases} \frac{1}{2}|u_i - u_j - a|^2 + \frac{1}{2}|u_j - u_i - a|^2 & \text{if } B = \{i, j\}, \\ V(u_i) & \text{if } B = \{i\}. \end{cases}$$

With this notation, (4) agrees with (21). Hence, condition **H1** is obvious, while **H2** follows because there are only a finite number of sets in $\mathcal{S}$ which intersect a finite set $X \subset \Lambda$. More precisely, if we set

$$c_R := 1 + \sharp\{i \in \mathbb{Z}^d \mid |i| \le R\},$$

we have that

$$\begin{aligned} &\sharp\{B \in \mathcal{S} \mid B \cap X \neq \emptyset\} \\ &\quad = \sharp\{i \in \mathbb{Z}^d \mid \{i\} \cap X \neq \emptyset\} \\ &\qquad + \sharp\{\{i, j\} \subset \mathbb{Z}^d \mid |i - j| \le R,\ \{i, j\} \cap X \neq \emptyset\} \\ &\quad \le \sharp X + \sum_{x \in X} \sharp\{i \in \mathbb{Z}^d \mid |i - x| \le R\} \\ &\quad = c_R \sharp X. \end{aligned} \tag{24}$$

Moreover, if $\gamma \in \mathbb{Z}^d$, we have that $\mathcal{T}_\gamma u(p) = u(p + \gamma)$ and so **H3** plainly follows.

The validity of **H4** is an easy consequence of (22). Condition **H6** follows from (23) and the facts that V is bounded and $F = \Lambda/G = \{0\}$.

Furthermore, recalling (6),

$$|||\sigma||| = \sup_{1 \le k \le d} |\omega_k|$$

and so

$$H_B(\sigma) \le |\omega \cdot (i - j) - a|^2 + \|V\|_{L^\infty} \le C(|||\sigma|||^2 + 1),$$

for a suitable $C > 0$ (possibly depending on R and a), thus yielding **H7**, via (24). Conditions **H8** and **H9** plainly follow by using (24). Condition **H10** is obviously fulfilled by taking $\delta(B)$ to be the diameter of B.

Finally, conditions **A1**–**A4** are satisfied in this case, due to Proposition 8. □

3 Proof of Theorem 7

The idea underneath the proof of Theorem 7 is the following. We introduce a reduced energy functional by subtracting the energy value at $u^{(0)}$ and identifying sets under the periodicity induced by H.

The reduction by periodicity is needed to make the reduced energy finite. However, in principle, such reduced energy might have different critical points than the original one. This is a somewhat delicate business, because the interactions are nonlocal and points inside a fundamental domain may feel the influence of points outside. On the other hand, conditions **A1**–**A4** will ensure that the variational argument patterned in [7, 8] may be adapted to the present case. Namely, we will follow any given interpolation $u^{(s)}$ of $u^{(0)}$ and $u^{(1)}$, with, say $s \in [0, 1]$, under a suitable "gradient flow" (or "heat flow"). We then study the "basin of attraction" of this flow assuming, by contradiction, that no other critical point exists inside the foliation gap. The contradiction will then be obtained by covering the interval [0, 1] with two disjoint open intervals, namely the basins of attraction of the points 0 and 1.

Let us now start with the details of the proof of Theorem 7.

Without loss of generality, we may suppose that $u^{(0)}$ and $u^{(1)}$ are at the edge of a gap in the (possibly singular) lamination induced by the minimizers given in Theorem 6. Indeed, if the lamination has no gaps between $u^{(0)}$ and $u^{(1)}$, then infinitely many leaves lie between $u^{(0)}$ and $u^{(1)}$, thus giving the claim of Theorem 7.

Thus, we are now going to construct a new solution inside the gap bordered by $u^{(0)}$ and $u^{(1)}$.

For this, the next result (which is very similar to Lemma 36 of [8]), will be important:

Lemma 10 *Let $u^{(0)}$ and $u^{(1)}$ be at the edge of a gap in the minimal (possibly singular) lamination of Theorem 6. Let $\tilde{H}$ be a normal subgroup of H so that $\sharp(H/\tilde{H}) < +\infty$. Then,*

$$\sum_{p\in\Lambda/\tilde{H}} |u^{(1)}(p) - u^{(0)}(p)| \le \sharp(\Lambda/H)\cdot\sharp(H/\tilde{H}). \tag{25}$$

The proof of Lemma 10 is obtained if we consider the sum involved in the left hand side of (25). We can divide the range of the sum according to the class Λ/H. Then, for each set of the classes we can perform a translation and a subtraction to reduce it to the fundamental domain. Then, by the assumption that $u^{(0)}$, $u^{(1)}$ are at the edge of the gaps of a foliation, we obtain that the gaps have to add to the volume of the fundamental domain, yielding the claim of Lemma 10 (see Lemma 36 of [8] for details).

We point out that, in the setting of Lemma 10, we always have that

$$\left|u^{(1)}(p) - u^{(0)}(p)\right| \le 1,$$

since $u^{(0)}$ and $u^{(1)}$ lie on the edge of a gap of the minimal lamination.

3.1 Some Compactness Properties

We consider the sup-norm

$$\|u\|_H := \sup_{p\in\Lambda/H} |u(p)|.$$

We define

$$\mathcal{C}_\sigma(u^{(0)}, u^{(1)}) := \{u \in \mathcal{B}_\sigma \mid u^{(0)} \le u \le u^{(1)}\},$$

where $u^{(0)}$ and $u^{(1)}$ are as in the statement of Theorem 7. To avoid unnecessary complications, we will use $\mathcal{C}$ as a short-hand notation for $\mathcal{C}_\sigma(u^{(0)}, u^{(1)})$ in the sequel.

It is easy to see (see, e.g., Lemma 37 in [8]) that the following compactness property holds:

Lemma 11 *Let $u_n \in \mathcal{C}$ for any $n \in \mathbb{N}$. Then, there exists a subsequence u_{n_k} and a configuration $u \in \mathcal{C}$ such that*

$$\lim_{k\to+\infty} u_{n_k}(p) = u(p), \tag{26}$$

for any $p \in \Lambda$.

The idea of the proof is that, because of the a-priori bounds on $u^{(1)} - u^{(0)}$ in (25), given any $\epsilon > 0$, the functions differ by more than ϵ only on a finite set (see, e.g., the proof of next result).

Pointwise and uniform convergence thus agree on $\mathcal{C}$, as next result points out:

Lemma 12 *If $u_n \in \mathcal{C}$ is such that*

$$\lim_{n\to+\infty} u_n(p) = u(p),$$

for any $p \in \Lambda$, then u_n converges to u in the sup*-norm $\|\cdot\|_H$.*

Proof Fix $\epsilon > 0$. By Lemma 10, there exists a finite set

$$X_\epsilon = \{p_1, \dots, p_m\} \subset \Lambda/H$$

in such a way that

$$|u_n(p) - u^{(0)}(p)| \le \epsilon,$$

for any $n \in \mathbb{N}$ and any $p \in (\Lambda/H) - X_\epsilon$. Since $\mathcal{C}$ is obviously closed under pointwise limits, $u \in \mathcal{C}$ and

$$|u(p) - u^{(0)}(p)| \le \epsilon,$$

for any $p \in (\Lambda/H) - X_\epsilon$.

Also, using again the pointwise convergence of u_n, we have that, for each $j = 1, \dots, m$, there exists $n(\epsilon, j)$ such that

$$|u_n(p_j) - u(p_j)| \le \epsilon,$$

for any $n \ge n(\epsilon, j)$. Let

$$n(\epsilon) := \max\{n(\epsilon, 1), \dots, n(\epsilon, m)\}.$$

Then,

$$|u_n(p) - u(p)| \le \epsilon,$$

for any $n \ge n(\epsilon)$ and any $p \in X_\epsilon$.

Thus, if $n \ge n(\epsilon)$,

$$\begin{aligned} \|u_n - u\|_H \le& \sup_{p\in X_\epsilon} |u_n(p) - u(p)| \\ &+ \sup_{p\in(\Lambda/H)-X_\epsilon} (|u_n(p) - u^{(0)}(p)| + |u^{(0)}(p) - u(p)|) \\ \le& \, 3\epsilon, \end{aligned}$$

proving the desired claim. □

As an easy consequence of Lemmata 11 and 12, we thus deduce that $\mathcal{C}$ is compact in the sup-norm:

Corollary 13 *Let $u_n \in \mathcal{C}$ for any $n \in \mathbb{N}$. Then, there exists a subsequence u_{n_k} and a configuration $u \in \mathcal{C}$ so u_{n_k} converges to u in the* sup*-norm $\|\cdot\|_H$.*

3.2 The Renormalized Reduced Energy

We now generalize an approach already used in [7]. That is, we introduce a *renormalized energy functional* and study its variational properties. The key point is that this renormalized energy has the same ground states and critical points of the original interaction. Nevertheless, it is given by a convergent and differentiable function. Hence, we can use the direct methods of calculus of variations. In our case, we will find it convenient to use the gradient flow of this renormalized energy.

In this section, we establish some properties of the renormalized energy. We will make clear the relation between the critical points and the ground states of the original problem and the analogous objects in the renormalized energy framework.

The main difference between the renormalized energy introduced here and that of [8] is that the one introduced here has to take into account the possibility that the cocycle contains some rational components, so we have to introduce some identifications.

We recall the equivalence relation introduced before **A3** and we define

$$\mathcal{S}_H := \{[B]_H \text{ with } B \in \mathcal{S}\}. \tag{27}$$

We observe that another way of looking at condition **A3** is by saying that

$$\begin{aligned}&\text{given any } p \in \Lambda \text{ and any } [B]_H \in \mathcal{S}_H,\\ &\text{there exists at most one } A \in [B]_H \text{ such that } A \ni p.\end{aligned} \tag{28}$$

We set

$$\tilde{\mathcal{C}}_{\sigma,H} = \{u \in \mathcal{O}_\sigma \mid \mathcal{T}_g u = \mathcal{R}_{\sigma(g)} u, \text{ for any } g \in H\}. \tag{29}$$

It is also convenient to define $\mathcal{C}_{\sigma,H}$ to be the set of all configurations $u \in \tilde{\mathcal{C}}_{\sigma,H}$ that agree with a configuration $\tilde{v} \in \mathcal{C}$ in Λ/H outside a finite set (such $\tilde{v}$ may depend on u: we recall that the definition of $\mathcal{C}$ was given at the beginning of Sect. 3.1).

Notice that if $u \in \mathcal{B}_\sigma$, then $\mathcal{T}_g u = \mathcal{R}_{\sigma(g)} u$ for any $g \in \mathcal{H}$, and so, since $\mathcal{H} \supset H$, we have that

$$\mathcal{C}_\sigma(u^{(0)}, u^{(1)}) = \mathcal{C} \subset \mathcal{B}_\sigma \subset \tilde{\mathcal{C}}_{\sigma,H}.$$

Concerning the definition in (27), we note that if $[B]_H \in \mathcal{S}_H$, then there exists $h \in H$ in such a way that $(hB) \cap (\Lambda/H) \neq \emptyset$, because Λ/H agrees with the fundamental domain.

For any $[B]_H \in \mathcal{S}_H$ and any $u \in \tilde{\mathcal{C}}_{\sigma,H}$, we also define

$$H_{[B]_H}(u) := H_B(u).$$

We remark that this is a good definition: indeed, if $[A]_H = [B]_H$, then $A = hB$ for some $h \in H$ and so

$$H_A(u) = H_{hB}(u) = H_B(\mathcal{T}_h u) = H_B(\mathcal{R}_{\sigma(h)} u) = H_B(u),$$

thanks to **H3**, **H4** and (27).

With this setting, given $u, v \in \tilde{\mathcal{C}}_{\sigma,H}$, we define

$$S_H(u, v) := \sum_{[B]_H \in \mathcal{S}_H} (H_{[B]_H}(u) - H_{[B]_H}(v)). \tag{30}$$

The renormalized energy function is then

$$S_H(u) := S_H(u, u^{(0)}),$$

where $u^{(0)}$ is as in the statement of Theorem 7.

We now deal with the convergence of the above series.

Lemma 14 *Let $u, v \in \tilde{\mathcal{C}}_{\sigma,H}$. Then, the following inequalities hold*:

$$\begin{aligned}|S_H(u,v)| &\le \sum_{[B]_H \in \mathcal{S}_H} |H_{[B]_H}(u) - H_{[B]_H}(v)| \\ &\le C \sum_{p \in \Lambda/H_{2k^\star}} |u_p - v_p|,\end{aligned}$$

where $C > 0$ here above is the finite quantity on the left hand side of (10) *and $k^\star$ was introduced in* **A2**.

Proof The first inequality is obvious. We thus use **A2** and **H8** to get the second one:

$$\begin{aligned}&\sum_{[B]_H \in \mathcal{S}_H} |H_{[B]_H}(u) - H_{[B]_H}(v)| \\ &\quad\le \sum_{[B]_H \in \mathcal{S}_H} \sum_{p \in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau v)) \right| d\tau |u_p - v_p| \\ &\quad\le \sum_{p \in \Lambda} \sum_{\substack{B \in \mathcal{S} \\ B \ni p \\ B \cap (\Lambda/H) \neq \emptyset}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau v)) \right| d\tau |u_p - v_p| \\ &\quad\le \sum_{p \in \Lambda/H_{2k^\star}} \sum_{\substack{B \in \mathcal{S} \\ B \ni p \\ B \cap (\Lambda/H) \neq \emptyset}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau v)) \right| d\tau |u_p - v_p|,\end{aligned}$$

which yields the claim. □

By Lemmata 10 and 14, we conclude that:

Corollary 15 *Let $u, v \in \mathcal{C}_{\sigma,H}$. Then, the series in* (30) *is absolutely convergent.*

Closely related to the above properties is that the renormalized energy is continuous with respect to the pointwise convergence.

Lemma 16 *Let $u_n \in \mathcal{C}$ be such that*

$$\lim_{n \to +\infty} u_n(p) = u(p),$$

for any $p \in \Lambda/H$. Then,

$$\lim_{n \to +\infty} \sum_{p \in \Lambda/H} |u_n(p) - u(p)| = 0, \tag{31}$$

$$\lim_{n\to+\infty} S_H(u_n) = S_H(u) \tag{32}$$

and

$$\lim_{n\to+\infty} \sum_{p\in\Lambda/H}\Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u_n)\Bigg)^2 = \sum_{p\in\Lambda/H}\Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u)\Bigg)^2. \tag{33}$$

Proof By Lemma 10, fixed $\epsilon > 0$, we can find a finite set $X_\epsilon \subset \Lambda/H$ in such a way that

$$\sum_{p\in(\Lambda/H)-X_\epsilon} |u^{(1)} - u^{(0)}| \le \epsilon.$$

Then,

$$\lim_{n\to+\infty} \sum_{p\in\Lambda/H} |u_n(p) - u(p)| \le \lim_{n\to+\infty} \sum_{p\in X_\epsilon} |u_n(p) - u(p)| + \epsilon = \epsilon.$$

By taking ϵ as small as we wish, we thus obtain the proof of (31).
Furthermore, using Corollary 15, **A2** and **H8**,

$$\begin{aligned}
&|S_H(u) - S_H(u_n)| \\
&\quad\le \sum_{[B]_H\in\mathcal{S}_H} |H_{[B]_H}(u) - H_{[B]_H}(u_n)| \\
&\quad\le \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{2k^\star})}} |H_B(u) - H_B(u_n)| \\
&\quad\le \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{2k^\star})}} \sum_{p\in B} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau u_n))\right| d\tau\, |u(p) - u_n(p)| \\
&\quad\le \sum_{p\in(\Lambda/H_{2k^\star})} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau u + (1-\tau u_n))\right| d\tau\, |u(p) - u_n(p)| \\
&\quad\le C \sum_{p\in(\Lambda/H_{2k^\star})} |u(p) - u_n(p)| \\
&\quad= C\sharp(H/H_{2k^\star}) \sum_{p\in(\Lambda/H)} |u(p) - u_n(p)|,
\end{aligned}$$

for a suitable $C > 0$. The latter estimate and (31) yield (32).
Moreover, by exploiting **A2** and **H8** once more, we conclude that

$$\begin{aligned}
&\left|\sum_{p\in\Lambda/H}\Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u_n)\Bigg)^2 - \sum_{p\in\Lambda/H}\Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u)\Bigg)^2\right| \\
&\quad\le \sum_{p\in\Lambda/H}\Bigg[\Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left|\frac{\partial H_B}{\partial u_p}(u_n)\right| + \left|\frac{\partial H_B}{\partial u_p}(u)\right|\Bigg)\cdot\Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left|\frac{\partial H_B}{\partial u_p}(u_n) - \frac{\partial H_B}{\partial u_p}(u)\right|\Bigg)\Bigg]
\end{aligned}$$

$$\leq 2C \sum_{p\in\Lambda/H} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{q\in B} \left(\int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau u_n + (1-\tau)u) \right| d\tau \cdot |u_n(q) - u(q)| \right)$$

$$= 2C \sum_{q\in\Lambda/H_{2k^\star}} \sum_{\substack{B\in\mathcal{S}\\ B\ni q}} \sum_{p\in B} \left(\int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau u_n + (1-\tau)u) \right| d\tau \cdot |u_n(q) - u(q)| \right)$$

$$\leq 2C^2 \sum_{q\in\Lambda/H_{2k^\star}} |u_n(q) - u(q)|$$

$$= 2C^2 \sharp(H/H_{2k^\star}) \sum_{q\in\Lambda/H} |u_n(q) - u(q)|,$$

for some $C > 0$. This completes the proof of (33), by exploiting (31) once more. □

We now deal with the variational properties of S_H.

First, we observe that it is clear that the critical points of the renormalized energy are the same as the critical points in the sense of Definition 5. The reason is that the renormalized energy contains exactly the same terms that involve u than the original energy. The additional terms that we subtract depend only on the fixed ground state we have chosen. They do not contribute to the variational problem even if they make the sum convergent.

Our aim will be to show that, if $v \in \tilde{\mathcal{C}}_{\sigma,H}$ is a ground state, than the fact that $u \in \tilde{\mathcal{C}}_{\sigma,H}$ is also a ground state is equivalent to $S_H(u,v) = 0$ (see Corollary 18 below). To achieve this goal, we use the following auxiliary result:

Lemma 17 *Let $u, v \in \mathcal{C}_{\sigma,H}$. Suppose that v is a ground state and that $u = v$ outside a finite set $X \subset \Lambda/H$. Then, $S_H(u,v) \geq 0$.*

Proof Given $n \in \mathbb{N}$ (to be taken large in the sequel), we now perturb u into v outside the fundamental domain Λ/H_n. More explicitly, we let

$$v_p^{(n)} := \begin{cases} u_p & \text{if } p \in \Lambda/H_n, \\ v_p & \text{if } p \notin \Lambda/H_n. \end{cases}$$

Let also

$$\tilde{Y} := \left(\bigcup_{h\in H} (hX) \right),$$

$$Y := (\Lambda/H_n) \cap \tilde{Y}.$$

By construction,

$$\sharp Y = \sharp(H/H_n)\sharp X,$$

thence Y is a finite subset of Λ. Moreover, $v^{(n)} = v$ outside Y. Consequently, since v is a ground state,

$$0 \leq \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset}} (H_B(v^{(n)}) - H_B(v))$$

$$= \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\subset(\Lambda/H_n)}} (H_B(v^{(n)}) - H_B(v)) + \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\not\subset(\Lambda/H_n)}} (H_B(v^{(n)}) - H_B(v))$$
$$= \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\subset(\Lambda/H_n)}} (H_B(u) - H_B(v)) + \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\not\subset(\Lambda/H_n)}} (H_B(v^{(n)}) - H_B(v)).$$

This inequality and **A2** yield that

$$0 \le \sum_{\substack{B\in\mathcal{S}\\ B\cap Y\neq\emptyset\\ B\subset(\Lambda/H_n)}} (H_B(u) - H_B(v)) + \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})}} |H_B(v^{(n)}) - H_B(v)|.$$

Since $u = v$ outside $\tilde{Y}$, the latter estimate reduces to

$$0 \le \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_n)}} (H_B(u) - H_B(v)) + \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})}} |H_B(v^{(n)}) - H_B(v)|. \tag{34}$$

We now observe that

$$\sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})}} |H_B(v^{(n)}) - H_B(v)|$$
$$\le \sum_{p\in\Lambda} \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})\\ B\ni p}} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau v^{(n)} + (1-\tau)v)\right| d\tau\, |v_p^{(n)} - v_p|$$
$$= \sum_{p\in(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})} \sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_{n+k^\star})-(\Lambda/H_{n-k^\star})\\ B\ni p}} \int_0^1 \left|\frac{\partial H_B}{\partial u_p}(\tau v^{(n)} + (1-\tau)v)\right| d\tau\, |v_p^{(n)} - v_p|$$
$$\le Cc(n) \sum_{p\in X} |u_p - v_p|, \tag{35}$$

by (10) and **A4**. Let now

$$a_n(B) := \sharp\{A \in [B]_H \text{ s.t. } A \subset \Lambda/H_n\}.$$

Note that

$$\sum_{\substack{B\in\mathcal{S}\\ B\subset(\Lambda/H_n)}} (H_B(u) - H_B(v)) = \sum_{[B]_H\in\mathcal{S}_H} a_n(B)(H_B(u) - H_B(v)). \tag{36}$$

Let also

$$b_n(B) := \frac{a_n(B)}{\sharp(H/H_n)}.$$

We deduce from (34), (35), (36) and Lemma 10 that

$$0 \leq \sum_{[B]_H \in \mathcal{S}_H} b_n(B)(H_{[B]_H}(u) - H_{[B]_H}(v)) + \frac{\tilde{C}c(n)}{\sharp(H/H_n)},$$

for a suitable $\tilde{C} > 0$ (possibly depending on u and v). We thus take the limit as $n \to +\infty$ and we deduce from **A4** and Corollary 15 that

$$0 \leq \sum_{[B]_H \in \mathcal{S}_H} (H_{[B]_H}(u) - H_{[B]_H}(v)),$$

as desired. □

Corollary 18 *Let $u, v \in \mathcal{C}_{\sigma,H}$. Suppose that v is a ground state. Then, $S_H(u,v) \geq 0$. Moreover, $S_H(u,v) = 0$ if and only if u is also a ground state.*

Proof Since $u, v \in \mathcal{C}_{\sigma,H}$, there exists a finite set $\bar{X} \subset \Lambda/H$ and two configurations $\bar{u}, \bar{v} \in \mathcal{C}$ in such a way that $u = \bar{u}$ and $v = \bar{v}$ outside $\bar{X}$.

Fix $\epsilon > 0$. By Lemma 10, we find a finite set $X_\epsilon \subset \Lambda$ in such a way that

$$\sum_{p \in (\Lambda/H_{2k^\star}) - X_\epsilon} |u_p^{(1)} - u_p^{(0)}| \leq \frac{\epsilon}{C}, \tag{37}$$

where C is the quantity in (10). Without loss of generality, $X_\epsilon \supset \bar{X}$.

We define

$$u_p^{(\epsilon)} := \begin{cases} u_p & \text{if } p \in X_\epsilon, \\ v_p & \text{if } p \notin X_\epsilon. \end{cases}$$

Therefore, using Corollary 15 and Lemmas 14 and 17, we get that

$$\begin{aligned} S_H(u,v) &= S_H(u, u^{(\epsilon)}) + S_H(u^{(\epsilon)}, v) \\ &\geq -C \sum_{p \in \Lambda/H_{2k^\star}} |u_p - u_p^{(\epsilon)}| + 0 \\ &= -C \sum_{p \in (\Lambda/H_{2k^\star}) - X_\epsilon} |u_p - u_p^{(\epsilon)}| \\ &\geq -C \sum_{p \in (\Lambda/H_{2k^\star}) - X_\epsilon} |u_p^{(1)} - u_p^{(0)}|. \end{aligned}$$

We conclude from (37) that $S_H(u,v) \geq -\epsilon$, and so $S_H(u,v) \geq 0$, since ϵ may be taken arbitrarily small. This proves the first claim.

Let us now suppose that both u and v are ground states. Then, by the first claim,

$$S_H(u,v) \geq 0 \quad \text{and} \quad S_H(v,u) \geq 0.$$

Since $S_H(u,v) + S_H(v,u) = 0$ thanks to Corollary 15, we conclude that $S_H(u,v) = 0$, as desired.

Let us now suppose that v is a ground state and that $S_H(u,v) = 0$. Let w be a configuration agreeing with u outside a finite set X. By **A1**, we have that $X \subset \Lambda/H_{n_0}$ for a suitable $n_0 \in \mathbb{N}$. Without loss of generality, we may assume that $n_0 \geq k^\star$. In fact, by **A2**, if $B \in \mathcal{S}$ is such that $B \cap X \neq \emptyset$, then $B \subset \Lambda/H_{n_1}$, with $n_1 := n_0 + k^\star$.

Thus, we may extend w outside Λ/H_{n_1} in such a way that, if we call $\tilde{w}$ such extension, we have that $\tilde{w} \in \mathcal{C}_{\sigma,H_{n_1}}$.

Then, since v is a ground state minimizer, we know from the first claim of Corollary 18 (applied here to $S_{H_{n_1}}$ instead of S_H) that $S_{H_{n_1}}(\tilde{w}, v) \geq 0$. Also,

$$S_{H_{n_1}}(u, v) = \sharp(H/H_{n_1}) S_H(u, v),$$

since both u and v are in $\mathcal{C}_{\sigma,H}$, and so $S_{H_{n_1}}(u, v) = 0$.

Hence, by Corollary 15,

$$S_{H_{n_1}}(\tilde{w}, u) = S_{H_{n_1}}(\tilde{w}, v) - S_{H_{n_1}}(u, v) \geq 0.$$

Accordingly,

$$\begin{aligned} 0 &\geq \sum_{[B]_{H_{n_1}} \in \mathcal{S}_{H_{n_1}}} (H_{[B]_H}(u) - H_{[B]_H}(\tilde{w})) \\ &= \sum_{\substack{B \in \mathcal{S} \\ B \cap X \neq \emptyset}} (H_B(u) - H_B(w)), \end{aligned}$$

thence u is a ground state. □

Remark 19 The introduction of the renormalized energy is rather standard as a heuristic method in Physics when one considers the energies of excitations in a background.

The energy of the background may not be finite, but if we consider only the renormalized energy generated by the excitation (i.e. the sum of the difference of the energy terms corresponding to the configuration and the ground state), we obtain a finite sum.

When one considers the Frenkel–Kontorova model, which is included in our general framework as indicated in Sect. 2 the renormalized energy is called Peierls–Nabarro energy of dislocations.

The Peierls–Nabarro barrier is the smallest renormalized energy of an equilibrium point. There are heuristic arguments which indicate that this Peierls–Nabarro barrier vanishes when (and only when) the dislocations are free to move. See [3]. Given the statements about convergence of the renormalized energy, we see that the Peierls–Nabarro barrier can be rigorously defined for the configurations in $\mathcal{C}$.

Using the remarks after Theorem 7, we see that either the lamination of minimizers is a full foliation of that there is a critical point which is not a ground state. By Corollary 18, the renormalized energy should be strictly positive.

Hence, we obtain as a corollary that the Peierls–Nabarro barrier vanishes if and only if the lamination of minimal sets does not have any gap.

For the case of one-dimensional problems with nearest neighbor interactions, this was established in [15].

3.3 Some Remarks on the Notion of Periodicity

Lemma 20 *For any $u \in \mathcal{O}_\sigma$, $s \in \mathbb{Z}$, $\gamma \in G$, $p \in B \in \mathcal{S}$, we have that*

$$\frac{\partial H_B}{\partial u_p}(\mathcal{R}_s u) = \frac{\partial H_B}{\partial u_p}(u) = \frac{\partial H_{\gamma^{-1}B}}{\partial u_{\gamma^{-1}p}}(\mathcal{T}_\gamma u).$$

Proof The first equality plainly follows from **H4**. The second one is a consequence of **H3** by means of the following argument. Since

$$H_{\gamma^{-1}B}(\mathcal{T}_\gamma u) = H_{\gamma(\gamma^{-1}B)}(u) = H_B(u),$$

we have that

$$\frac{\partial H_B}{\partial u_p}(u) = \sum_{q\in\gamma^{-1}B} \frac{\partial H_{\gamma^{-1}B}}{\partial u_q}(\mathcal{T}_\gamma u)\frac{\partial(\mathcal{T}_\gamma u)_q}{\partial u_p}$$

$$= \sum_{q\in\gamma^{-1}B} \frac{\partial H_{\gamma^{-1}B}}{\partial u_q}(\mathcal{T}_\gamma u)\frac{\partial u_{\gamma q}}{\partial u_p} = \frac{\partial H_{\gamma^{-1}B}}{\partial u_{\gamma^{-1}p}}(\mathcal{T}_\gamma u),$$

as desired. □

Corollary 21 *If $u \in \tilde{\mathcal{C}}_{\sigma,H}$ and $h \in H$,*

$$\frac{\partial H_B}{\partial u_p}(u) = \frac{\partial H_{hB}}{\partial u_{hp}}(u).$$

Proof Since $\gamma := h^{-1} \in H \subset \mathcal{H}$, we have that $s := \sigma(\gamma) \in \mathbb{Z}$, and so we use Lemma 20 twice to gather that

$$\frac{\partial H_B}{\partial u_p}(u) = \frac{\partial H_{hB}}{\partial u_{hp}}(\mathcal{T}_\gamma u) = \frac{\partial H_{hB}}{\partial u_{hp}}(\mathcal{R}_s u) = \frac{\partial H_{hB}}{\partial u_{hp}}(u),$$

as desired. □

Recalling (28), we now introduce the following notation: given $p \in \Lambda$ and $[B]_H \in \mathcal{S}_H$, we denote by B_p the unique (if any) set $A \in [B]_H$ such that $A \ni p$. With this setting, we get the following result:

Corollary 22 *Let $u \in \tilde{\mathcal{C}}_{\sigma,H}$ and $v \in \mathcal{O}_\sigma$. Suppose that*

$$\mathcal{T}_h v = v \tag{38}$$

for any $h \in H$. Then,

$$\sum_{p\in B} \frac{\partial H_B}{\partial u_p}(u) v_p = \sum_{p\in\Lambda/H} \frac{\partial H_{B_p}}{\partial u_p}(u) v_p \tag{39}$$

for any $B \in \mathcal{S}$. Also, the above are sums of finite number of terms.

Proof That only a finite number of terms are involved in the sum in the left hand side of (39) follows from the fact that $\sharp B < +\infty$ for any $B \in \mathcal{S}$ (recall the definition of $\mathcal{S}$ given in Sect. 1.2.2).

Concerning the sum on the right hand side of (39), suppose that there are $p_1, \ldots, p_\ell \in \Lambda/H$ in such a way B_{p_j} is defined, i.e., $h_j B \ni p_j$ for some $h_j \in H$, with $j = 1, \ldots, \ell$. Let $q_j := h_j^{-1} p_j$. Then, all the q_j's are different if so are the p_j's, since the p_j's lie in the fundamental domain Λ/H.

By construction, all the q_j's lie in B and so $\ell \le \sharp B < +\infty$, which shows that the sum in the right hand side of (39) contains only finitely many terms.

We now prove (39). Given $p \in B$, we write $p = hq$, with $h \in H$ and $q \in \Lambda/H$. Such a q is obviously unique, because Λ/H is a fundamental domain and the uniqueness of h is here a consequence of **G3**. Thence, the notation $p = hq$ with $h \in H$ and $q \in \Lambda/H$ is uniquely determined.

Then, we use (38) and Corollary 21 to gather that

$$\sum_{p\in B} \frac{\partial H_B}{\partial u_p}(u) v_p = \sum_{\substack{h\in H\\ q\in\Lambda/H\\ hq\in B}} \frac{\partial H_B}{\partial u_{hq}}(u) v_{hq} = \sum_{\substack{h\in H\\ q\in(\Lambda/H)\cap(h^{-1}B)}} \frac{\partial H_{h(h^{-1}B)}}{\partial u_{hq}}(u) v_q$$
$$= \sum_{\substack{h\in H\\ q\in(\Lambda/H)\cap(h^{-1}B)}} \frac{\partial H_{h^{-1}B}}{\partial u_q}(u) v_q. \tag{40}$$

Recalling **A3**, we conclude that the union in

$$\bigcup_{h\in H} (hB)$$

is disjoint and so the latter term in (40) equals

$$\sum_{q\in\Lambda/H} \frac{\partial H_{B_q}}{\partial u_q}(u)\, v_q,$$

which gives the desired result. □

3.4 The Heat Flow

We now consider the solution of a "heat flow" problem, which, indeed, is the "gradient flow" of our renormalized energy functional.

Given $u \in \mathcal{C}$ and $t > 0$, we denote by $\Phi^t(u)$ the configuration $p \mapsto U(p,t)$ that satisfies

$$\begin{cases} \partial_t U(p,t) = -\sum\limits_{\substack{B\in\mathcal{S}\\ B\ni p}} \dfrac{\partial H_B}{\partial u_p}(U(\cdot,t)) & \text{for any } p \in \Lambda \text{ and } t > 0,\\ U(p,0) = u(p) & \text{for any } p \in \Lambda. \end{cases} \tag{41}$$

Notice that the series in (41) converges, thanks to **H8**. Moreover, the following properties hold:

Lemma 23 *If $u \in \mathcal{C}$, then Φ^t is well-defined for any $t > 0$. Also, $\Phi^t(\mathcal{C}) \subset \mathcal{C}$,*

$$\mathcal{T}_h \Phi^t(u) = \mathcal{R}_{\sigma(h)} \Phi^t(u), \tag{42}$$

$$\partial_t(\mathcal{T}_h \Phi^t(u)) = \partial_t \Phi^t(u) \tag{43}$$

for any $h \in H$, and

$$-\frac{d}{dt} S_H(\Phi^t(u)) = \sum_{p\in\Lambda/H} \Bigg(\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(\Phi^t(u)) \Bigg)^2 \ge 0. \tag{44}$$

Furthermore, if $u \in \mathcal{C}$, the map $t \mapsto \Phi^t(u)$ belongs to $C([0,+\infty),\mathcal{C}) \cap C^1((0,+\infty),\mathcal{C})$.

Moreover, the following result on the dependence on initial data holds: fixed any $T > 0$, the map

$$\mathcal{C} \ni u \mapsto \Phi^t(u)$$

is continuous in the ℓ^∞ topology and its modulus of continuity in the ℓ^∞ topology is uniform for any $t \in [0,T]$.

Proof Fixed any $p \in \Lambda$,

$$\begin{aligned}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \left| \frac{\partial H_B}{\partial u_p}(u) - \frac{\partial H_B}{\partial u_p}(v) \right| &\le \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \sum_{q\in B} \int_0^1 \left| \frac{\partial^2 H_B}{\partial u_p \partial u_q}(\tau u + (1-\tau)v) \right| d\tau \cdot \|u-v\|_H \\ &\le C\|u-v\|_H,\end{aligned}$$

for any $u, v \in \mathcal{C}$, due to **H8**. Accordingly, the map

$$\mathcal{C} \ni u \longmapsto \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \frac{\partial H_B}{\partial u_p}(u)$$

is uniformly Lipschitz continuous with respect to the sup-norm. Then, the short time existence of the heat flow with continuous dependence on the initial data follows from the theory of ODE's in Banach spaces (see, e.g., [11]). The short time flow may be continued provided that it does not exit from $\mathcal{C}$ (see, e.g., the Prolongation Lemma in [11]).

Using the ferromagnetism assumption **H9**, it is possible to show that the gradient flow satisfies a comparison principle and that it preserves the order among configurations.

We recall that the (weak) comparison principle, i.e., the fact that $u \ge v$ yields $\Phi^t(u) \ge \Phi^t(v)$ for any $t \ge 0$, is a consequence of the ferromagnetic property **H9**. The proof just uses that the ferromagnetic property immediately implies that $\frac{d}{dt} D\Phi^t = M(t) D\Phi^t$ where M is a matrix with off diagonal entries which are non-negative. Furthermore, the operator M is the limit of finite dimensional truncations. For details see, e.g., Lemma 4.3 in [12]. See also [9] for a sharper version in a particular case. We remark that this is the only place in this paper where we use the ferromagnetism assumption.

Since the set $\mathcal{C}$ is defined by the order properties and by the translations, which commute with the flow, we conclude that $\Phi^t(\mathcal{C}) \subset \mathcal{C}$ and since we have uniform bounds for the time of definition for all the initial data in $\mathcal{C}$, we conclude, applying the Prolongation Lemma in [11] that the flow Φ^t in $\mathcal{C}$ is defined for all time.

We now prove (42) and (43). In fact, since (43) plainly follows from (42), we focus on the proof of (42). To this end, if $U(\cdot,t) = \Phi^t(u)$, we have that

$$U(hp,0) = u(hp) = u(p) + \sigma(h) = U(p,0) + \sigma(h)$$

since $u \in \mathcal{C}$, and

$$\begin{aligned}-\partial_t U(hp,t) &= \sum_{\substack{B\in\mathcal{S}\\ h^{-1}B\ni p}} \frac{\partial H_{h(h^{-1}B)}}{\partial u_{hp}}(U(\cdot,t)) = \sum_{\substack{B\in\mathcal{S}\\ h^{-1}B\ni p}} \frac{\partial H_{h^{-1}B}}{\partial u_p}(U(\cdot,t)) \\ &= \sum_{\substack{A\in\mathcal{S}\\ A\ni p}} \frac{\partial H_A}{\partial u_p}(U(\cdot,t)+\sigma(h)) = -\partial_t(U(p,t)+\sigma(h)),\end{aligned}$$

where Lemma 20 and Corollary 21 have been utilized. Therefore, by the uniqueness of the solution of the Cauchy problem, $U(hp,t)=U(p,t)+\sigma(h)$, yielding (42).

It now remains to prove (44). To this extent, let $t>0$ and $\delta>0$ (the case $\delta<0$ with $t+\delta>0$ is analogous). We observe that (38) is fulfilled here with $v:=\partial_t(\Phi^t(u))$, thanks to (43). Therefore, we use Corollary 22 and (44) to obtain that

$$\begin{aligned}\frac{d}{ds}H_{[B]_H}(\Phi^{t+\delta s}(u)) &= \frac{d}{ds}H_B(\Phi^{t+\delta s}(u))\\ &= \delta\sum_{p\in B}\frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta s}(u))\partial_\tau(\Phi^\tau(u))_p|_{\tau=t+\delta s}\\ &= -\delta\sum_{p\in\Lambda/H}\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t+\delta s}(u))\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_A}{\partial u_p}(\Phi^{t+\delta s}(u)).\end{aligned}$$

In the last term here above, the first sum consists only in a finite numbers of terms, due to Corollary 22, and the second sum is absolutely convergent, due to **H8**. Therefore, for any $B\in\mathcal{S}$,

$$\begin{aligned}&\frac{H_{[B]_H}(\Phi^{t+\delta}(u))-H_{[B]_H}(\Phi^t(u))}{\delta}\\ &\quad=\frac{1}{\delta}\int_0^1\frac{d}{ds}H_{[B]_H}(\Phi^{t+\delta s}(u))\,ds\\ &\quad=-\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t+\delta s}(u))\frac{\partial H_{A_p}}{\partial u_p}(\Phi^{t+\delta s}(u))\,ds\\ &\quad=-\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_{B_p}}{\partial u_p}(\Phi^t(u))\frac{\partial H_{A_p}}{\partial u_p}(\Phi^t(u))+R_{\delta,B},\end{aligned}\tag{45}$$

where **A3** has been used to replace A with A_p and $R_{\delta,B}$ satisfies

$$\begin{aligned}|R_{\delta,B}|\le\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\int_0^1&\left|\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t+\delta s}(u))\right|\left|\frac{\partial H_{A_p}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{A_p}}{\partial u_p}(\Phi^t(u))\right|\\ &+\left|\frac{\partial H_{A_p}}{\partial u_p}(\Phi^t(u))\right|\left|\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{B_p}}{\partial u_p}(\Phi^t(u))\right|ds.\end{aligned}\tag{46}$$

Moreover, for any $B\in\mathcal{S}$,

$$\begin{aligned}&\left|\frac{\partial H_B}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\right|\\ &\quad\le\sum_{q\in B}\int_0^1\left|\frac{\partial^2 H_B}{\partial u_p\partial u_q}(\tau\Phi^{t+\delta s}(u)+(1-\tau)\Phi^t(u))\right||(\Phi^{t+\delta s}(u)-\Phi^t(u))_q|\,d\tau.\end{aligned}$$

Accordingly, from (46),

$$\begin{aligned}
&\sum_{[B]_H\in\mathcal{S}_H}|R_{\delta,B}|\\
&\quad\le\sum_{[B]_H\in\mathcal{S}_H}\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\A\ni p}}\int_0^1\left|\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t+\delta s}(u))\right|\left|\frac{\partial H_{A_p}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{A_p}}{\partial u_p}(\Phi^{t}(u))\right|\\
&\qquad+\left|\frac{\partial H_{A_p}}{\partial u_p}(\Phi^{t}(u))\right|\left|\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{B_p}}{\partial u_p}(\Phi^{t}(u))\right|ds\\
&\quad\le\sum_{p\in\Lambda/H}\sum_{\substack{B\in\mathcal{S}\\B\ni p}}\sum_{\substack{A\in\mathcal{S}\\A\ni p}}\int_0^1\left|\frac{\partial H_{B}}{\partial u_p}(\Phi^{t+\delta s}(u))\right|\left|\frac{\partial H_{A}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{A}}{\partial u_p}(\Phi^{t}(u))\right|\\
&\qquad+\left|\frac{\partial H_{A}}{\partial u_p}(\Phi^{t}(u))\right|\left|\frac{\partial H_{B}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{B}}{\partial u_p}(\Phi^{t}(u))\right|ds\\
&\quad\le 2C_1\sum_{p\in\Lambda/H}\sum_{\substack{B\in\mathcal{S}\\B\ni p}}\int_0^1\left|\frac{\partial H_{B}}{\partial u_p}(\Phi^{t+\delta s}(u))-\frac{\partial H_{B}}{\partial u_p}(\Phi^{t}(u))\right|ds\\
&\quad\le 2C_1\sum_{p\in\Lambda/H}\sum_{\substack{B\in\mathcal{S}\\B\ni p}}\sum_{q\in B}\int_0^1\int_0^1\left|\frac{\partial^2 H_{B}}{\partial u_p\partial u_q}(\tau\Phi^{t+\delta s}(u)+(1-\tau)\Phi^{t}(u))\right|\\
&\qquad\times|(\Phi^{t+\delta s}(u)-\Phi^{t}(u))_q|\,d\tau\,ds\\
&\quad\le 2C_1C_2\sum_{q\in\Lambda/H_{2k^\star}}\int_0^1|(\Phi^{t+\delta s}(u)-\Phi^{t}(u))_q|\,ds,
\end{aligned}\tag{47}$$

where $C_1>0$ (resp., $C_2>0$) is the first (resp., the second) derivative bound given by **H8**, and **A2** has also been used.

Note also that

$$\lim_{\delta\to 0}|(\Phi^{t+\delta s}(u)-\Phi^{t}(u))_q|=0\tag{48}$$

and that

$$|(\Phi^{t+\delta s}(u)-\Phi^{t}(u))_q|\le u_q^{(1)}-u_q^{(0)}\tag{49}$$

for any $q\in\Lambda$, since we have already shown that $\Phi^t(\mathcal{C})\subset\mathcal{C}$. Also, from Lemma 10,

$$\sum_{q\in\Lambda/H_{2k^\star}}\int_0^1 u_q^{(1)}-u_q^{(0)}\,ds<+\infty.\tag{50}$$

The Dominated Convergence Theorem, (47), (48), (49) and (50) imply that

$$\lim_{\delta\to 0}\sum_{[B]_H\in\mathcal{S}_H}|R_{\delta,B}|=0.$$

Consequently, from Corollary 15, (45), and **A2**,

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{S_H(\Phi^{t+\delta}(u)) - S_H(\Phi^t(u))}{\delta}\\
&\quad = \lim_{\delta\to 0}\sum_{[B]_H\in\mathcal{S}_H}\frac{H_{[B]_H}(\Phi^{t+\delta}(u)) - H_{[B]_H}(\Phi^t(u))}{\delta}\\
&\quad = \lim_{\delta\to 0} - \sum_{[B]_H\in\mathcal{S}_H}\sum_{p\in\Lambda/H}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_{B_p}}{\partial u_p}(\Phi^t(u))\frac{\partial H_{A_p}}{\partial u_p}(\Phi^t(u)) + \sum_{[B]_H\in\mathcal{S}_H} R_{\delta,B}\\
&\quad = -\sum_{p\in\Lambda/H}\sum_{\substack{B\in\mathcal{S}\\ B\ni p}}\sum_{\substack{A\in\mathcal{S}\\ A\ni p}}\frac{\partial H_B}{\partial u_p}(\Phi^t(u))\frac{\partial H_A}{\partial u_p}(\Phi^t(u)) + 0.
\end{aligned}$$

From this, (44) plainly follows, thus completing the proof of Lemma 23. □

Remark 24 There are natural strengthening of hypothesis **H9** which lead to the a strong comparison principle. That is, $u \geq v$ and $u \neq v$ imply $\Phi^t(u)_i > \Phi^t(v)_i$ for all $i \in \Lambda$ and all $t > 0$.

This follows from the matrix of $D\Phi_t(u)$ being strictly positive for all u. See the discussions in [5, 9, 12]. For example, this is satisfied in the one dimensional Frenkel–Kontorova models or on the models with uniform twist.

Once one has a strong comparison principle for solutions, it follows that the critical points satisfy a strong maximum principle, so that the singular laminations of critical points are indeed laminations.

Given any $s \in [0, 1]$, we now consider the linear interpolation between $u^{(0)}$ and $u^{(1)}$, by defining

$$u^{(s)} := (1-s)\,u^{(0)} + su^{(1)}.$$

It is easily seen that $u^{(s)} \in \mathcal{C}$. Furthermore, the following compactness result holds:

Corollary 25 *Fixed* $s \in [0, 1]$, *there exists a sequence* $t_{n,s}$ *and a configuration* $u_\star^{(s)} \in \mathcal{C}$, *which is a solution of our variational equation* (12), *such that*

$$\lim_{n\to+\infty} t_{n,s} = +\infty,$$

$$\lim_{n\to+\infty} \|\Phi^{t_{n,s}}(u^{(s)}) - u_\star^{(s)}\|_H = 0.$$

Proof Let $f(t) := S_H(\Phi^t(u))$. By Lemmata 15 and 23, we know that f is decreasing and bounded by below. Thus,

$$\lim_{t\to+\infty} f(t) = \inf_{[0,\infty)} f.$$

Then, there must exists a divergent sequence of times $t_{n,s}$ such that

$$\lim_{n\to+\infty} f'(t_{n,s}) = 0. \tag{51}$$

If not, one would have that $f'(t) \leq -c$ for some $c > 0$ and any t large enough, but then

$$0 = \lim_{t \to +\infty} f(t+1) - f(t) \leq -c,$$

which is a contradiction.

Moreover, by Lemma 23 and Corollary 13, we may suppose, possibly taking subsequences, that $\Phi^{t_{n,s}}(u^{(s)})$ converges as $t_{n,s} \to +\infty$ in the sup-norm $\| \cdot \|_H$, say to a suitable $u_\star^{(s)} \in \mathcal{C}$. By Lemma 16, (51), (44) and (12), we have that $u_\star^{(s)}$ is a solution of our variational equation (12). □

3.5 The Basin of Attraction of the Heat Flow

The rest of the proof of Theorem 7 is now very close to the variational argument of [7, 8]. We provide full details for the reader's facility.

We consider the "basin of attraction" of the configurations $u^{(0)}$ and $u^{(1)}$ with respect to the heat flow Φ^t. Namely, for $i \in \{0, 1\}$, we define

$$\beta_i := \left\{ s \in [0,1] \,\Big|\, \lim_{t \to +\infty} \|\Phi^t(u^{(s)}) - u^{(i)}\|_H = 0 \right\}. \tag{52}$$

Note that, since $u^{(i)}$ is a solution of our variational equation (12), then $\Phi^t(u^{(i)}) = u^{(i)}$, and so $i \in \beta_i$, for $i \in \{0, 1\}$. We also consider the configurations in $\mathcal{C}$ with zero renormalized energy, that is the set

$$\mu_\sigma := \{u \in \mathcal{C} \mid S_H(u) = 0\}.$$

We now introduce the set of the solutions of our variational equation which belong to $\mathcal{C}$ and whose S_H-value is less than r. More precisely, given $r > 0$, we define

$$\nu_\sigma(r) := \{u \in \mathcal{C} \mid u \text{ is a solution of our variational equation (12) and } S_H(u) \leq r\}.$$

By Lemma 15, we have that

$$u^{(0)}, u^{(1)} \in \mu_\sigma \subset \nu_\sigma(r),$$

for any $r > 0$. We now investigate the properties of the basins of attraction when μ_σ and $\nu_\sigma(r)$ are "as small as possible".

Lemma 26 *Assume that*

$$\mu_\sigma - \{u^{(0)}, u^{(1)}\} = \emptyset. \tag{53}$$

Then, there exists $r_0 > 0$ in such a way that the following holds. Let $r > 0$ and suppose that there exists $\gamma \in C([0,1], \mathcal{C})$ such that $\gamma(0) = u^{(0)}$, $\gamma(1) = u^{(1)}$ and $S_H(\gamma(t)) \leq r$ for any $t \in [0,1]$. Then, $r \geq r_0$.

Proof We argue by contradiction. Suppose that, for any $n \in \mathbb{N}$, there exists

$$\gamma_n \in C([0,1], \mathcal{C}) \tag{54}$$

such that

$$\gamma_n(0) = u^{(0)}, \qquad \gamma_n(1) = u^{(1)}, \tag{55}$$

$$S_H(\gamma_n(t)) \leq 1/n \quad \text{for any } t \in [0,1]. \tag{56}$$

Let us define

$$g_n(t) := \|\gamma_n(t) - u^{(0)}\|_H.$$

Then, $g_n \in C([0,1])$, thanks to (54). Also, $g_n(0) = 0$ and $g_n(1) = \|u^{(1)} - u^{(0)}\|_H$, due to (55). Thus, there exists $t_n \in [0,1]$ in such a way

$$g_n(t_n) = \frac{\|u^{(1)} - u^{(0)}\|_H}{2} =: c.$$

That is,

$$\|\gamma_n(t_n) - u^{(0)}\|_H = c. \tag{57}$$

We recall that $c > 0$, because $u^{(0)} \not\equiv u^{(1)}$.

Moreover, by Corollary 13, perhaps taking subsequences, we have that $\gamma_n(t_n)$ converges in the sup-norm to some $\bar{u} \in \mathcal{C}$. By (57), $\|\bar{u} - u^{(0)}\|_H = c$ and thus $\bar{u} \notin \{u^{(0)}, u^{(1)}\}$. Thus, by (53), $S_H(\bar{u}) > 0$. On the other hand, by (56) and Lemma 16, $S_H(\bar{u}) = 0$. This contradiction yields the desired result. □

We remark the fact that r_0 in Lemma 26 is independent of the path γ.

Lemma 27 *Assume that*

$$\nu_\sigma(r_0) - \{u^{(0)}, u^{(1)}\} = \emptyset, \tag{58}$$

where r_0 is as in Lemma 26. Then, there exists $r_1 > 0$ such that the following holds. If $v \in \mathcal{C}$ and $\|v - u^{(0)}\|_H \le r_1$, then

$$\lim_{t\to+\infty} \|\Phi^t(v) - u^{(0)}\|_H = 0.$$

Analogously, if $v \in \mathcal{C}$ and $\|v - u^{(1)}\|_H \le r_1$, then

$$\lim_{t\to+\infty} \|\Phi^t(v) - u^{(1)}\|_H = 0.$$

Proof We prove the first assertion, the second one following analogously. The idea of the proof is the following: let us start the flow close to $u^{(0)}$; we know by Corollary 25 that this flow has to converge to a solution of our variational equation, up to subsequence. Since the flow decreases the renormalized energy, (58) yields that any subsequence of the above flow approaches either $u^{(0)}$ or $u^{(1)}$. If all the subsequences approaches $u^{(0)}$ we are done, thence we have to exclude the existence of subsequences approaching $u^{(1)}$. But if any of such subsequences existed, a path joining $u^{(0)}$ and $u^{(1)}$ with small energy would be constructed, in contradiction with Lemma 26.

We now give the proof in full detail. The proof is by contradiction: assume that there exists $v \in \mathcal{C}$ such that

$$\|v - u^{(0)}\|_H \le r_1 \tag{59}$$

in such a way

$$\Phi^t(v) \text{ does not converge to } u^{(0)} \text{ in the norm } \|\cdot\|_H. \tag{60}$$

Here, r_1 will be a suitably small quantity, defined as follows. By Lemma 10, we can find a finite set $X_{r_0} \subset \Lambda/H_{2k^\star}$ in such a way that

$$\sum_{p\in(\Lambda/H_{2k^\star})-X_{r_0}} |u_p^{(1)} - u_p^{(0)}| \le \frac{r_0}{4C}, \tag{61}$$

where C is the quantity in (10). We then set

$$r_1 := \min\left\{ r_0, \frac{r_0}{4C\sharp X_{r_0}\sharp(H/H_{2k^\star})} \right\}.$$

We claim that

$$\sup_{\substack{w\in\mathcal{C}\\ \|w-u^{(0)}\|_H\le r_1}} S_H(w) \le \frac{r_0}{2}. \tag{62}$$

Indeed, if w is as here above, we get from **H8** and (61) that

$$\begin{aligned}
S_H(w) &\le \sum_{\substack{B\in\mathcal{S}\\ B\subset\Lambda/H_{2k^\star}}} \sum_{p\in B} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau w + (1-\tau)u^{(0)}) \right| d\tau \cdot |w_p - u_p^{(0)}| \\
&\le \sum_{p\in\Lambda/H_{2k^\star}} \sum_{\substack{B\in\mathcal{S}\\ B\ni p}} \int_0^1 \left| \frac{\partial H_B}{\partial u_p}(\tau w + (1-\tau)u^{(0)}) \right| d\tau \cdot |u_p^{(1)} - u_p^{(0)}| \\
&\le C \sum_{p\in\Lambda/H_{2k^\star}} |u_p^{(1)} - u_p^{(0)}| \\
&\le C \sum_{p\in X_{r_0}} |u_p^{(1)} - u_p^{(0)}| + \frac{r_0}{4} \\
&\le C\|u^{(1)} - u^{(0)}\|_H \sharp X_{r_0}\sharp(H/H_{2k^\star}) + \frac{r_0}{4} \\
&\le \frac{r_0}{2},
\end{aligned}$$

thus proving (62).

From this and (59),

$$S_H(v) \le \frac{r_0}{2}, \tag{63}$$

provided that r_1 is small enough with respect to r_0.

Also, recalling Corollary 25, possibly extracting a subsequence, we have that $\Phi^{t_n}(v)$ converges to a suitable configuration $\bar{u} \in \mathcal{C}$ in the norm $\|\cdot\|_H$, and that

$$\bar{u} \text{ is a solution of our variational equation (12).} \tag{64}$$

Furthermore, it follows from (63) and Lemma 23 that

$$S_H(\Phi^t(v)) \le \frac{r_0}{2} \quad \text{for any } t \ge 0. \tag{65}$$

Consequently, by Lemma 16,

$$S_H(\bar{u}) \leq \frac{r_0}{2}. \tag{66}$$

Note that $\bar{u}$ cannot be equal to $u^{(0)}$, because of (60); thus, (64), (66) and (58) imply that $\bar{u} = u^{(1)}$.

Therefore, there exists $T > 0$ such that

$$\|\Phi^T(v) - u^{(1)}\|_H = \|\Phi^T(v) - \bar{u}\|_H \leq r_1. \tag{67}$$

Let now

$$\tilde{\gamma}(t) := \Phi^{2T(t-1/4)}(v), \tag{68}$$

for any $t \in [1/4, 3/4]$.

Then, $\tilde{\gamma} \in C([1/4, 3/4], \mathcal{C})$, thanks to Lemma 23. Also, $\tilde{\gamma}(1/4) = v$ and $\tilde{\gamma}(3/4) = \Phi^T(v)$. Let

$$\gamma(t) := \begin{cases} u^{(0)} + 4t(v - u^{(0)}) & \text{if } t \in [0, 1/4), \\ \tilde{\gamma}(t) & \text{if } t \in [1/4, 3/4], \\ \Phi^T(v) + (4t-3)(u^{(1)} - \Phi^T(v)) & \text{if } t \in (3/4, 1]. \end{cases}$$

Then,

$$\gamma \in C([0,1], \mathcal{C}), \qquad \gamma(0) = u^{(0)}, \qquad \gamma(1) = u^{(1)}. \tag{69}$$

Moreover, if $t \in [0, 1/4)$,

$$\|\gamma(t) - u^{(0)}\|_H \leq \|v - u^{(0)}\|_H \leq r_1,$$

due to (59), and, if $t \in (3/4, 1]$

$$\|\gamma(t) - u^{(1)}\|_H \leq \|\Phi^T(v) - u^{(1)}\|_H \leq r_1,$$

due to (67). Therefore, making use of (62), we gather that $S_H(\gamma(t)) \leq r_0/2$, for $t \in [0, 1/4) \cup (3/4, 1]$, provided that r_1 is small enough. In fact, recalling (65) and (68),

$$S_H(\gamma(t)) \leq r_0/2 \quad \text{for any } t \in [0, 1]. \tag{70}$$

But the existence of a path γ satisfying (69) and (70) is in contradiction with Lemma 26. Notice indeed that the assumptions of Lemma 26 are fulfilled here, since (58) implies (53).

This contradiction ends the proof of Lemma 27. □

Lemma 28 *Assume* (58). *Then,* β_0 *and* β_1, *as defined in* (52), *are open in the relative topology of* $[0, 1]$.

Proof We prove that β_0 is open, the argument for β_1 being analogous. Let $\bar{s} \in \beta_0$ and take

$$s \in [0, 1] \cap [\bar{s} - \epsilon, \bar{s} + \epsilon].$$

We show that $s \in \beta_0$, provided that $\epsilon > 0$ is sufficiently small. For this, assume, by contradiction, that $s \notin \beta_0$. Then, there exists a sequence $t_n \to +\infty$ in such a way

$$\Phi^{t_n}(u^{(s)}) \text{ does not converge to } u^{(0)} \text{ in the norm } \|\cdot\|_H. \tag{71}$$

Now, since $\bar{s} \in \beta_0$, there exists $\bar{n} \in \mathbb{N}$ such that

$$\|\Phi^{t_{\bar{n}}}(u^{(\bar{s})}) - u^{(0)}\|_H \leq \frac{r_1}{2},$$

where r_1 is the quantity given in Lemma 27. By the continuity with respect to the initial data (recall Lemma 23), we thus have that

$$\|\Phi^{t_{\bar{n}}}(u^{(s)}) - u^{(0)}\|_H \leq r_1,$$

provided that ϵ is small enough.

Therefore, by Lemma 27, it follows that $\Phi^t(\Phi^{t_{\bar{n}}}(u^{(s)}))$ converges to $u^{(0)}$ in the norm $\|\cdot\|_H$. That is, $\Phi^{t_n}(u^{(s)})$ converges to $u^{(0)}$ in the norm $\|\cdot\|_H$. Since the latter assertion is in contradiction with (71), the proof of the desired result is complete. □

Corollary 29 *Suppose that* (58) *holds. Then, there exists* $s \in [0, 1] - (\beta_0 \cup \beta_1)$.

Proof Assume, by contradiction, that $[0, 1] = \beta_0 \cup \beta_1$. Let

$$\bar{s} := \sup\{s \in \beta_0\}. \tag{72}$$

By Lemma 28, β_0 and β_1 are nonempty, disjoint, relatively open subsets of $[0, 1]$. Since $1 \in \beta_1$ and β_1 is open, we have that

$$\bar{s} < 1. \tag{73}$$

Analogously, since $0 \in \beta_0$ and β_0 is open, we have that

$$\bar{s} > 0. \tag{74}$$

Thus, in the light of (72), there are two cases to distinguish: either $\bar{s} \in \beta_0$ or $\bar{s} \in \beta_1$.

If $\bar{s} \in \beta_0$, then, since β_0 is relatively open, it follows from (73) that $\bar{s} + \epsilon \in \beta_0$, provided that $\epsilon > 0$ is sufficiently small. But this contradicts (72).

If, on the other hand, $\bar{s} \in \beta_1$, the fact that β_1 is open and (74) imply that $\bar{s} - \epsilon \in \beta_1$, for any $\epsilon > 0$ sufficiently small. But this again contradicts (72).

These contradictions give the proof of Corollary 29. □

3.6 End of the Proof of Theorem 7

We distinguish two alternatives: either (58) is violated or it does hold.

If (58) is violated, take

$$u \in \nu_\sigma(r_0) - \{u^{(0)}, u^{(1)}\}.$$

Then, by construction, $u \in \mathcal{B}_\sigma$, u is a solution of our variational equation (12), $u^{(0)} \leq u \leq u^{(1)}$ and u does not coincide with either $u^{(0)}$ or $u^{(1)}$. Thus, u fulfills the claim of Theorem 7.

If, on the other hand, (58 holds), then there exists

$$s \in [0, 1] - (\beta_0 \cup \beta_1),$$

thanks to Corollary 29. Therefore, by Corollary 13, there exists a sequence $t_n \to +\infty$ in such a way that $\Phi^{t_n}(u^{(s)})$ converges in the norm $\|\cdot\|_H$ to a suitable $u \in \mathcal{C}$, which does not coincide with either $u^{(0)}$ or $u^{(1)}$. Then, the fact that $u \in \mathcal{C}$ gives that $u \in \mathcal{B}_\sigma$, and that $u^{(0)} \leq u \leq u^{(1)}$. Also, Corollary 25 implies that u is a solution of our variational equation (12). Hence u fulfills the claim of Theorem 7 in this case too.

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